

# Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials

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## Abstract

The well-posedness of a system of partial differential equations and dynamic boundary conditions, both of Cahn–Hilliard type, is discussed. The existence of a weak solution and its continuous dependence on the data are proved using a suitable setting for the conservation of a total mass in the bulk plus the boundary. A very general class of double-well like potentials is allowed. Moreover, some further regularity is obtained to guarantee the strong solution.

**Key words:** Cahn–Hilliard system, dynamic boundary condition, mass conservation, well-posedness, strong solution.

**AMS (MOS) subject classification:** 35K61, 35K25, 35D30, 35D35, 80A22.

## 1 Introduction

The Cahn–Hilliard equation [5, 14] yields a famous description of the evolution phenomena on the solid-solid phase separation. In general, an evolution process goes on diffusively. However, the phenomenon of the solid-solid phase separation does not seem to follow on this structure: more precisely, each phase concentrates and this process is usually known as spinodal decomposition. The Cahn–Hilliard equation is a celebrated model which describes this decomposition by the simple framework of partial differential equations. Thereon, the volume conservation is a key property of the structure: you can observe the pattern formation that is restricted by the property of conservation on the decomposition. On the other hand, in the real world there are many phenomena of pattern formation which

do not have the structure of conservation. However, there is a possibility that actually the structure of conservation is hidden somewhere, and we only cannot find it at the level of observation.

In this paper, a system coupling the same kind of equations and dynamic boundary conditions of Cahn–Hilliard type is investigated. We aim to describe it at once. Let  $0 < T < +\infty$  be some fixed time and let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ , be a bounded smooth domain occupied by a material: the boundary  $\Gamma$  of  $\Omega$  is supposed to be smooth enough as well. We start from the following equations of Cahn–Hilliard type in the domain  $Q := \Omega \times (0, T)$

$$\partial_t u - \Delta \mu = 0 \quad \text{in } Q, \quad (1.1)$$

$$\mu = -\Delta u + W'(u) - f \quad \text{in } Q, \quad (1.2)$$

where  $\partial_t$  denotes the partial derivative with respect to time and, as usual,  $\Delta$  represents the Laplacian operator acting on the space variables. Here, the unknowns  $u$  and  $\mu : Q \rightarrow \mathbb{R}$  stand for the order parameter and the chemical potential, respectively. In order to consider the dynamics on the boundary  $\Sigma := \Gamma \times (0, T)$ , we also introduce the unknowns  $u_\Gamma, \mu_\Gamma : \Sigma \rightarrow \mathbb{R}$  such that

$$u_\Gamma = u|_\Gamma, \quad \mu_\Gamma = \mu|_\Gamma \quad \text{on } \Sigma, \quad (1.3)$$

$u|_\Gamma$  and  $\mu|_\Gamma$  being the traces of  $u$  and  $\mu$ , and consider the same type of equations on the boundary

$$\partial_t u_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma, \quad (1.4)$$

$$\mu_\Gamma = \partial_\nu u - \Delta_\Gamma u_\Gamma + W'_\Gamma(u_\Gamma) - f_\Gamma \quad \text{on } \Sigma, \quad (1.5)$$

where the extra terms in (1.4) and (1.5) contain the outward normal derivative  $\partial_\nu$  on  $\Gamma$ , and where  $\Delta_\Gamma$  denotes the Laplace–Beltrami operator on  $\Gamma$  (see, e.g., [19, Chapter 3]). We can say that this kind of dynamic boundary condition (1.4)–(1.5) is a sort of transmission problem between the dynamics in the bulk  $\Omega$  and the one on the boundary  $\Gamma$ . Together with the conditions

$$u(0) = u_0 \quad \text{in } \Omega, \quad (1.6)$$

$$u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma, \quad (1.7)$$

the initial and boundary value problem expressed in (1.1)–(1.7) is termed (P). In (1.2) and (1.5), the nonlinear terms  $W'$  and  $W'_\Gamma$  play some important role, since they are the derivatives of the functions  $W$  and  $W_\Gamma$  usually referred as double-well potentials, with two minima and a local unstable maximum in between. The prototype model is provided by  $W(r) = W_\Gamma(r) = (1/4)(r^2 - 1)^2$  so that  $W'(r) = W'_\Gamma(r) = r^3 - r$ ,  $r \in \mathbb{R}$ , is the sum of an increasing function with a power growth and another smooth (in particular, Lipschitz continuous) function which breaks the monotonicity properties of the former and is related to the non-convex part of the potential  $W$  or  $W_\Gamma$ . To our knowledge, this problem (P) was formulated by Goldstein, Miranville and Schimperna [17] and analyzed from various viewpoints (see [7–9, 18]). In this paper, we treat more general cases for

such nonlinearities, that is, we let  $W'$  and  $W'_\Gamma$  be the sum of a maximal monotone graph and of a Lipschitz perturbation and we are able to show the existence of strong solutions for our system under appropriate assumptions. Our treatment is related to the approach followed in [6, 12] for some other class of problems.

As is well known, for the usual Cahn–Hilliard system (1.1)–(1.2), the conservation of (the mean value of)  $u$  is guaranteed under the homogeneous Neumann boundary condition

$$\partial_\nu \mu = 0 \quad \text{on } \Sigma,$$

for  $\mu$ ; namely, thanks to this, by simply integrating (1.1) over  $\Omega$  we easily obtain from (1.6) that

$$\int_{\Omega} u(t) dx = \int_{\Omega} u_0 dx \quad \text{for all } t \in [0, T].$$

Let us mention the related papers [1, 7, 21, 22] on which the structure of conservation was treated in an abstract framework. The new issue of the problem (P) is the natural consequence of a mass constraint involving the values of  $u$  both in the bulk and on the boundary. In fact, it arises as an outcome of (1.1) and (1.4) that the solution  $u$  satisfies

$$\int_{\Omega} u(t) dx + \int_{\Gamma} u_{\Gamma}(t) d\Gamma = \int_{\Omega} u_0 dx + \int_{\Gamma} u_{0\Gamma} d\Gamma \quad \text{for all } t \in [0, T]. \quad (1.8)$$

Concerning the model, let us point out that, under suitable choices of  $W'$  and  $W'_\Gamma$ , we could flexibly realize some dynamics for pattern formation respecting (1.8). For example, we may examine the case when every pattern is going to disappear, namely the bulk comes to be occupied by a single phase except near the boundary (occupied by another phase) and thus (1.8) plays as a conservation law. We invite the reader to compare the approach of this paper with the one adopted in [10], where the Allen–Cahn equation, coupled with dynamic boundary conditions, is investigated under a mass constraint which involves the solution inside the domain and its trace on the boundary. In that case, the constraint is rather imposed (in opposition with (1.8), which is a gift of the problem) and the system of nonlinear partial differential equations can be formulated as a variational inequality.

A brief outline of the present paper along with a short description of the various items is as follows.

In Section 2, we present the main results, consisting in the well-posedness of the system (1.1)–(1.7) of partial differential equations and dynamic boundary conditions, both of Cahn–Hilliard type. We define a weak and strong solution of the problem (P). We also write the system as an evolution inclusion.

In Section 3, we prove the continuous dependence on the data and this result entails the uniqueness property.

In Section 4, we prove the existence result. The proof is split in several steps. First, we construct an approximate solution by substituting the maximal monotone graphs with their Yosida regularizations. The solvability of the approximate problem is guaranteed by the abstract theory of doubly nonlinear evolution inclusions [13]. Moreover, arguing in a similar way as in [11], we show that the solution satisfies suitable regularity properties and uniform estimates. Finally, from these estimates, we can pass to the limit and conclude

the existence proof of the weak solution. Next, we can proceed by considering some additional uniform estimates in order to obtain the strong solution.

Finally, Section 5 contains an Appendix collecting some useful verifications.

Anyway, for the reader's convenience, a detailed index of sections and subsections follows.

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## 2 Main results

In this section, our main result is stated. We present our system of equations and conditions now:

$$\partial_t u - \Delta \mu = 0 \quad \text{a.e. in } Q, \quad (2.1)$$

$$\mu = -\Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \quad (2.2)$$

$$u_\Gamma = u|_\Gamma, \quad \mu_\Gamma = \mu|_\Gamma, \quad \partial_t u_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{a.e. on } \Sigma, \quad (2.3)$$

$$\mu_\Gamma = \partial_\nu u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma, \quad (2.4)$$

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma, \quad (2.5)$$

where  $f : Q \rightarrow \mathbb{R}$ ,  $f_\Gamma : \Sigma \rightarrow \mathbb{R}$ ,  $u_0 : \Omega \rightarrow \mathbb{R}$ ,  $u_{0\Gamma} : \Gamma \rightarrow \mathbb{R}$  are given functions;  $\beta$  stands for the subdifferential of the convex part  $\widehat{\beta}$  and  $\pi$  stands for the derivative of the concave perturbation  $\widehat{\pi}$  of a double well potential  $W(r) = \widehat{\beta}(r) + \widehat{\pi}(r)$  for all  $r \in \mathbb{R}$ . Here  $\beta$  is generalized to the case of maximal monotone graph in  $\mathbb{R} \times \mathbb{R}$ .  $\beta_\Gamma$  and  $\pi_\Gamma$  have the same property as  $\beta$  and  $\pi$ , respectively. Typical examples of  $\beta$ ,  $\beta_\Gamma$  and  $\pi$ ,  $\pi_\Gamma$  are given as follows:

- $\beta(r) = \beta_\Gamma(r) = r^3$ ,  $\pi(r) = \pi_\Gamma(r) = -r$  for all  $r \in \mathbb{R}$  with  $D(\beta) = D(\beta_\Gamma) = \mathbb{R}$  for the prototype double well potential  $W(r) = (r^2 - 1)^2/4$ ;

- $\beta(r) = \beta_\Gamma(r) = \ln((1+r)/(1-r))$ ,  $\pi(r) = \pi_\Gamma(r) = -2cr$  for all  $r \in D(\beta)$  with  $D(\beta) = D(\beta_\Gamma) = (-1, 1)$  for the logarithmic double well potential  $W(r) = ((1+r) \ln(1+r) + (1-r) \ln(1-r)) - cr^2$  where  $c > 0$  is a large constant which breaks convexity;
- $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r)$ ,  $\pi(r) = \pi_\Gamma(r) = -r$  for all  $r \in D(\beta)$  with  $D(\beta) = D(\beta_\Gamma) = [-1, 1]$  for the singular potential  $W(r) = I_{[-1,1]}(r) - r^2/2$ .

Of course, it is not necessary that  $\beta$  and  $\beta_\Gamma$  are the same graph or the same kind of graphs, what is important is that they respect the compatibility condition (A6), which is stated below. Our working assumption is that the boundary potential somehow dominates the potential in the bulk, cf. [6, 10–12] for analogous approaches. We also point out that a mixing of the first two cases is considered by the results reported in [7]: there, the inclusions in (2.2) and (2.4) actually reduce to the equalities  $\xi = \beta(u)$  and  $\xi_\Gamma = \beta_\Gamma(u_\Gamma)$ . In the present paper, we can handle also the case of effective graphs.

## 2.1 Weak formulation

We treat the problem (P) by a system of variational formulations. To this aim, we introduce the spaces  $H := L^2(\Omega)$ ,  $V := H^1(\Omega)$ ,  $H_\Gamma := L^2(\Gamma)$ ,  $V_\Gamma := H^1(\Gamma)$  with usual norms  $|\cdot|_H$ ,  $|\cdot|_V$ ,  $|\cdot|_{H_\Gamma}$ ,  $|\cdot|_{V_\Gamma}$  and inner products  $(\cdot, \cdot)_H$ ,  $(\cdot, \cdot)_V$ ,  $(\cdot, \cdot)_{H_\Gamma}$ ,  $(\cdot, \cdot)_{V_\Gamma}$ , respectively. Moreover, we put  $\mathbf{H} := H \times H_\Gamma$  and

$$\mathbf{V} := \{(z, z_\Gamma) \in V \times V_\Gamma : z_\Gamma = z|_\Gamma\}.$$

Then,  $\mathbf{H}$  and  $\mathbf{V}$  are Hilbert spaces with the inner products

$$\begin{aligned} (\mathbf{u}, \mathbf{z})_{\mathbf{H}} &:= (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \mathbf{u} := (u, u_\Gamma), \mathbf{z} := (z, z_\Gamma) \in \mathbf{H}, \\ (\mathbf{u}, \mathbf{z})_{\mathbf{V}} &:= (u, z)_V + (u_\Gamma, z_\Gamma)_{V_\Gamma} \quad \text{for all } \mathbf{u} := (u, u_\Gamma), \mathbf{z} := (z, z_\Gamma) \in \mathbf{V}, \end{aligned}$$

and related norms. As a remark, let us restate that if  $\mathbf{z} := (z, z_\Gamma) \in \mathbf{V}$  then  $z_\Gamma$  is exactly the trace of  $z$  on  $\Gamma$ ; while, if  $\mathbf{z} := (z, z_\Gamma)$  is just in  $\mathbf{H}$ , then  $z \in H$  and  $z_\Gamma \in H_\Gamma$  are independent. From now on, we use the notation of a bold letter like  $\mathbf{u}$  to denote the pair which corresponds to the letter, that is  $(u, u_\Gamma)$  for  $\mathbf{u}$ . Now, for  $\mathbf{z} := (z, z_\Gamma) \in \mathbf{V}$  and  $t \in (0, T)$ , we test (2.1) by  $z$  and use (2.3) to infer

$$\int_\Omega \partial_t u(t) z dx + \int_\Gamma \partial_t u_\Gamma(t) z_\Gamma d\Gamma + \int_\Omega \nabla \mu(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma \mu_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma = 0. \quad (2.6)$$

We also test (2.2) by  $z$  and exploit (2.4); then we obtain

$$\begin{aligned} & \int_\Omega \mu(t) z dx + \int_\Gamma \mu_\Gamma(t) z_\Gamma d\Gamma \\ &= \int_\Omega \nabla u(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma + \int_\Omega (\xi(t) + \pi(u(t)) - f(t)) z dx \\ & \quad + \int_\Gamma (\xi_\Gamma(t) + \pi_\Gamma(u_\Gamma(t)) - f_\Gamma(t)) z_\Gamma d\Gamma. \end{aligned} \quad (2.7)$$

Now, let us take  $\mathbf{z} = \mathbf{1} := (1, 1)$  in (2.6) and integrate with respect to time getting

$$\int_{\Omega} u(t) dx + \int_{\Gamma} u_{\Gamma}(t) d\Gamma = m_0(|\Omega| + |\Gamma|) \quad \text{for all } t \in [0, T],$$

where  $|\Omega| := \int_{\Omega} 1 dx$ ,  $|\Gamma| := \int_{\Gamma} 1 d\Gamma$  and

$$m_0 := \frac{\int_{\Omega} u_0 dx + \int_{\Gamma} u_{0\Gamma} d\Gamma}{|\Omega| + |\Gamma|}$$

is a sort of mean value for our problem. Then the mean value of the variable  $\mathbf{u}$  is conserved in the sense that

$$m(\mathbf{u}(t)) = m(\mathbf{u}_0) = m_0 \quad \text{for all } t \in [0, T],$$

where

$$m(\mathbf{z}) := \frac{\int_{\Omega} z dx + \int_{\Gamma} z_{\Gamma} d\Gamma}{|\Omega| + |\Gamma|} \quad \text{for all } \mathbf{z} \in \mathbf{H}. \quad (2.8)$$

The duality pairing between  $\mathbf{V}^*$  and  $\mathbf{V}$  is denoted by  $\langle \cdot, \cdot \rangle_{\mathbf{V}^*, \mathbf{V}}$  and it is understood that  $\mathbf{H}$  is embedded in  $\mathbf{V}^*$  in the usual way, i.e., such that  $\langle \mathbf{u}, \mathbf{z} \rangle_{\mathbf{V}^*, \mathbf{V}} = (\mathbf{u}, \mathbf{z})_{\mathbf{H}}$  for all  $\mathbf{u} \in \mathbf{H}$  and  $\mathbf{z} \in \mathbf{V}$ . Then, with the help of the later Remark 2 (see also the comments in [22, pp. 5674–5675]), we can rewrite (2.6) as

$$\langle \mathbf{u}'(t), \mathbf{z} \rangle_{\mathbf{V}^*, \mathbf{V}} + a(\boldsymbol{\mu}(t), \mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in \mathbf{V},$$

where  $\mathbf{u}'(t)$  denotes now the time derivative of the vectorial function and the bilinear form  $a(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$  is defined by

$$a(\mathbf{u}, \mathbf{z}) := \int_{\Omega} \nabla u \cdot \nabla z dx + \int_{\Gamma} \nabla_{\Gamma} u_{\Gamma} \cdot \nabla_{\Gamma} z_{\Gamma} d\Gamma \quad \text{for all } \mathbf{u}, \mathbf{z} \in \mathbf{V}.$$

We also introduce the subspace  $\mathbf{H}_0$  of  $\mathbf{H}$  by

$$\mathbf{H}_0 := \{ \mathbf{z} \in \mathbf{H} : m(\mathbf{z}) = 0 \},$$

and  $\mathbf{V}_0 := \mathbf{V} \cap \mathbf{H}_0$  with their norms:  $|\mathbf{z}|_{\mathbf{H}_0} := |\mathbf{z}|_{\mathbf{H}}$  for all  $\mathbf{z} \in \mathbf{H}_0$  and

$$|\mathbf{z}|_{\mathbf{V}_0} := \sqrt{a(\mathbf{z}, \mathbf{z})} \quad \text{for all } \mathbf{z} \in \mathbf{V}_0.$$

Let us define the linear bounded operator  $\mathbf{F} : \mathbf{V}_0 \rightarrow \mathbf{V}_0^*$  by

$$\langle \mathbf{F}\mathbf{z}, \tilde{\mathbf{z}} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} := a(\mathbf{z}, \tilde{\mathbf{z}}), \quad \mathbf{z}, \tilde{\mathbf{z}} \in \mathbf{V}_0, \quad (2.9)$$

as well. Then we see that there exists  $c_p > 0$  such that

$$c_p |\mathbf{z}|_{\mathbf{V}}^2 \leq \langle \mathbf{F}\mathbf{z}, \mathbf{z} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} = |\mathbf{z}|_{\mathbf{V}_0}^2 \quad \text{for all } \mathbf{z} \in \mathbf{V}_0; \quad (2.10)$$

this is checked in the Appendix. Therefore, thanks to the fact  $|\mathbf{z}|_{\mathbf{V}_0}^2 \leq |\mathbf{z}|_{\mathbf{V}}^2$  for all  $\mathbf{z} \in \mathbf{V}_0$ , we see that  $|\cdot|_{\mathbf{V}_0}$  and  $|\cdot|_{\mathbf{V}}$  are equivalent norm on  $\mathbf{V}_0$  and then  $\mathbf{F}$  is the duality mapping from  $\mathbf{V}_0$  to  $\mathbf{V}_0^*$ . Additionally, we can define the inner product in  $\mathbf{V}_0^*$  by

$$(\mathbf{z}_1^*, \mathbf{z}_2^*)_{\mathbf{V}_0^*} := \langle \mathbf{z}_1^*, \mathbf{F}^{-1} \mathbf{z}_2^* \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} \quad \text{for all } \mathbf{z}_1^*, \mathbf{z}_2^* \in \mathbf{V}_0^*.$$

Then, we obtain  $\mathbf{V}_0 \hookrightarrow \mathbf{H}_0 \hookrightarrow \mathbf{V}_0^*$  (this is also checked in the Appendix), where “ $\hookrightarrow$ ” stands for the dense and compact embedding, namely  $(\mathbf{V}_0, \mathbf{H}_0, \mathbf{V}_0^*)$  is a standard Hilbert triplet.

## 2.2 Definition of the solution and main theorems

In order to define our solution we use the following additional notation: the variable  $\mathbf{v} := \mathbf{u} - m_0 \mathbf{1}$  with initial value  $\mathbf{v}_0 := \mathbf{u}_0 - m_0 \mathbf{1}$ , namely,  $(v, v_\Gamma) = (u - m_0, u_\Gamma - m_0)$  and  $(v_0, v_{0\Gamma}) = (u_0 - m_0, u_{0\Gamma} - m_0)$ ; the datum  $\mathbf{f} := (f, f_\Gamma)$ ; the nonlinearity  $\boldsymbol{\pi}(\mathbf{z}) := (\pi(z), \pi_\Gamma(z_\Gamma))$  for  $\mathbf{z} = (z, z_\Gamma) \in \mathbf{H}$ ; the further space  $\mathbf{W} := H^2(\Omega) \times H^2(\Gamma)$ .

The solution is defined as follows.

**Definition 2.1.** *The triplet  $(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\xi})$  is called the weak solution of (P) if*

$$\mathbf{v} \in H^1(0, T; \mathbf{V}_0^*) \cap L^\infty(0, T; \mathbf{V}_0) \cap L^2(0, T; \mathbf{W}),$$

$$\boldsymbol{\mu} \in L^2(0, T; \mathbf{V}),$$

$$\boldsymbol{\xi} = (\xi, \xi_\Gamma) \in L^2(0, T; \mathbf{H}), \quad \xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma$$

and they satisfy

$$\langle \mathbf{v}'(t), \mathbf{z} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} + a(\boldsymbol{\mu}(t), \mathbf{z}) = 0 \quad \text{for all } \mathbf{z} \in \mathbf{V}_0, \quad (2.11)$$

$$(\boldsymbol{\mu}(t), \mathbf{z})_{\mathbf{H}} = a(\mathbf{v}(t), \mathbf{z}) + (\boldsymbol{\xi}(t) + \boldsymbol{\pi}(\mathbf{v}(t) + m_0 \mathbf{1}) - \mathbf{f}(t), \mathbf{z})_{\mathbf{H}} \quad \text{for all } \mathbf{z} \in \mathbf{V}, \quad (2.12)$$

for a.a.  $t \in (0, T)$ , and

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \mathbf{H}_0. \quad (2.13)$$

**Remark 1.** Thanks to the regularity  $\mathbf{v} \in L^2(0, T; \mathbf{W})$ , we see that (2.12) implies that

$$\begin{aligned} \mu &= -\Delta u + \xi + \pi(u) - f \quad \text{a.e. in } Q, \\ \mu_\Gamma &= \partial_\nu u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma \quad \text{a.e. on } \Sigma, \end{aligned}$$

with  $u = v + m_0$  and  $u_\Gamma = v_\Gamma + m_0$ .

Next, we introduce the notion of *strong solution*: we ask the reader to let us use the variable  $\mathbf{u}$  (instead of  $\mathbf{v}$ ) here.

**Definition 2.2.** *The triplet  $(\mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$  is called the strong solution of (P) if*

$$\mathbf{u} \in W^{1,\infty}(0, T; \mathbf{V}^*) \cap H^1(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{W}),$$

$$\boldsymbol{\mu} \in L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; \mathbf{W}),$$

$$\boldsymbol{\xi} \in L^\infty(0, T; \mathbf{H}),$$

and they satisfy

$$\partial_t u - \Delta \mu = 0 \quad \text{a.e. in } Q, \quad (2.14)$$

$$\xi \in \beta(u), \quad \mu = -\Delta u + \xi + \pi(u) - f \quad \text{a.e. in } Q, \quad (2.15)$$

$$u_\Gamma = u|_\Gamma, \quad \mu_\Gamma = \mu|_\Gamma, \quad \partial_t u_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{a.e. on } \Sigma, \quad (2.16)$$

$$\xi_\Gamma \in \beta_\Gamma(u_\Gamma), \quad \mu_\Gamma = \partial_\nu u - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma \quad \text{a.e. on } \Sigma, \quad (2.17)$$

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma. \quad (2.18)$$

The first result states the continuous dependence on the data. The uniqueness of the component  $\mathbf{v}$  of the solution is also guaranteed by this theorem. We assume that

(A1)  $\mathbf{f} \in L^2(0, T; \mathbf{H})$ ;

(A2)  $\mathbf{u}_0 := (u_0, u_{0\Gamma}) \in \mathbf{V}$ ;

(A3)  $\beta, \beta_\Gamma$ , maximal monotone graphs in  $\mathbb{R} \times \mathbb{R}$ , are the subdifferentials

$$\beta = \partial\widehat{\beta}, \quad \beta_\Gamma = \partial\widehat{\beta}_\Gamma$$

of some proper lower semicontinuous and convex functions  $\widehat{\beta}$  and  $\widehat{\beta}_\Gamma : \mathbb{R} \rightarrow [0, +\infty]$  satisfying  $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$  with some effective domains  $D(\beta)$  and  $D(\beta_\Gamma)$ , respectively. This implies that  $0 \in \beta(0)$  and  $0 \in \beta_\Gamma(0)$ ;

(A4)  $\pi, \pi_\Gamma : \mathbb{R} \rightarrow \mathbb{R}$  are Lipschitz continuous functions with Lipschitz constants  $L$  and  $L_\Gamma$ , respectively;

Then, we obtain the following continuous dependence on the data.

**Theorem 2.1.** *Assume (A1)–(A4). For  $i = 1, 2$  let  $(\mathbf{v}^{(i)}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\xi}^{(i)})$  be a weak solution of (P) corresponding to the data  $\mathbf{f}^{(i)}$  and  $\mathbf{v}_0^{(i)}$ . Then, there exists a positive constant  $C > 0$ , depending only on  $c_p, L, L_\Gamma$  and  $T$ , such that*

$$\begin{aligned} & |\mathbf{v}^{(1)}(t) - \mathbf{v}^{(2)}(t)|_{\mathbf{V}_0^*}^2 + \int_0^t |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0}^2 \\ & \leq C \left\{ |\mathbf{v}_0^{(1)} - \mathbf{v}_0^{(2)}|_{\mathbf{V}_0^*}^2 + \int_0^t |\mathbf{f}^{(1)}(s) - \mathbf{f}^{(2)}(s)|_{\mathbf{V}^*}^2 ds \right\} \quad \text{for all } t \in [0, T]. \end{aligned} \quad (2.19)$$

The second result deals with the existence of the weak solution. To the aim, we further assume that:

(A5) either  $\mathbf{f} \in W^{1,1}(0, T; \mathbf{H})$  or  $\mathbf{f} \in L^2(0, T; \mathbf{V})$ ;

(A6)  $D(\beta_\Gamma) \subseteq D(\beta)$  and there exist positive constants  $\varrho, c_0 > 0$  such that

$$|\beta^\circ(r)| \leq \varrho |\beta_\Gamma^\circ(r)| + c_0 \quad \text{for all } r \in D(\beta_\Gamma); \quad (2.20)$$

(A7)  $m_0 \in \text{int}D(\beta_\Gamma)$  and the compatibility conditions  $\widehat{\beta}(u_0) \in L^1(\Omega)$ ,  $\widehat{\beta}_\Gamma(u_{0\Gamma}) \in L^1(\Gamma)$  hold.

The minimal section  $\beta^\circ$  of  $\beta$  is specified by  $\beta^\circ(r) := \{r^* \in \beta(r) : |r^*| = \min_{s \in \beta(r)} |s|\}$  and same definition applies to  $\beta_\Gamma^\circ$ . These assumptions are the same as in [6, 10, 12].

**Theorem 2.2.** *Under the assumptions (A2)–(A7), there exists a weak solution of the problem (P).*

About the strong solution, we refer the reader to Subsection 4.4.



## 2.3 Abstract formulation

In this subsection, an abstract formulation of the problem is given. We can write the problem as an evolution equation including a subdifferential operator: here, one can find some analogies with the approach followed in [10, 11, 21, 22].

We define the lower semicontinuous and convex functional  $\varphi : \mathbf{H}_0 \rightarrow [0, +\infty]$  by

$$\varphi(\mathbf{z}) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 dx + \frac{1}{2} \int_{\Gamma} |\nabla_{\Gamma} z_{\Gamma}|^2 d\Gamma & \text{if } \mathbf{z} \in \mathbf{V}_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then we claim that the subdifferential  $\partial\varphi$  on  $\mathbf{H}_0$  fulfills  $\partial\varphi(\mathbf{z}) = (-\Delta z, \partial_{\nu} z - \Delta_{\Gamma} z_{\Gamma})$  for  $\mathbf{z} \in D(\partial\varphi) = \mathbf{W} \cap \mathbf{V}_0$ : this is checked precisely in the Appendix (see Lemma C). We also note that (cf. (2.9)–(2.10))

$$2\varphi(\mathbf{z}) = a(\mathbf{z}, \mathbf{z}) = \langle \mathbf{F}\mathbf{z}, \mathbf{z} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} = |\mathbf{z}|_{\mathbf{V}_0}^2 \quad \text{for all } \mathbf{z} \in \mathbf{V}_0, \quad (2.21)$$

by collecting in this formula part of our notation.

At this point, we emphasize that our problem is equivalent to the following Cauchy problem for a suitable evolution equation.

$$\mathbf{v}'(t) + \mathbf{F}(\mathbf{P}\boldsymbol{\mu}(t)) = \mathbf{0} \quad \text{in } \mathbf{V}_0^*, \text{ for a.a. } t \in (0, T), \quad (2.22)$$

$$\boldsymbol{\mu}(t) = \partial\varphi(\mathbf{v}(t)) + \boldsymbol{\xi}(t) + \boldsymbol{\pi}(\mathbf{u}(t)) - \mathbf{f}(t) \quad \text{in } \mathbf{H}, \text{ for a.a. } t \in (0, T), \quad (2.23)$$

$$\boldsymbol{\xi}(t) \in \boldsymbol{\beta}(\mathbf{u}(t)) \quad \text{in } \mathbf{H}, \text{ for a.a. } t \in (0, T), \quad (2.24)$$

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \mathbf{H}_0, \quad (2.25)$$

where  $\mathbf{u} = \mathbf{v} + m_0 \mathbf{1} = (v + m_0, v_{\Gamma} + m_0)$  and  $\boldsymbol{\beta}(\mathbf{z}) := (\beta(z), \beta_{\Gamma}(z_{\Gamma}))$ ,

$$\mathbf{P}\mathbf{z} := \mathbf{z} - m(\mathbf{z})\mathbf{1} = (z - m(\mathbf{z}), z_{\Gamma} - m(\mathbf{z})), \quad (2.26)$$

with  $m(\mathbf{z})$  defined by (2.8), for all  $\mathbf{z} = (z, z_{\Gamma}) \in \mathbf{H}$ . Note that the projection operator  $\mathbf{P}$  acts as linear bounded operator both from  $\mathbf{V}$  to  $\mathbf{V}_0$  and from  $\mathbf{H}$  to  $\mathbf{H}_0$ . Moreover, it is easy to see that

$$\begin{aligned} (\mathbf{z}^*, \mathbf{P}\tilde{\mathbf{z}})_{\mathbf{H}_0} &= \left\{ \int_{\Omega} z^* \tilde{z} dx + \int_{\Gamma} z_{\Gamma}^* \tilde{z}_{\Gamma} \right\} - m(\tilde{\mathbf{z}}) \left\{ \int_{\Omega} z^* dx + \int_{\Gamma} z_{\Gamma}^* \right\} \\ &= (\mathbf{z}^*, \tilde{\mathbf{z}})_{\mathbf{H}} \quad \text{for all } \mathbf{z}^* \in \mathbf{H}_0 \text{ and } \tilde{\mathbf{z}} \in \mathbf{H}. \end{aligned} \quad (2.27)$$

**Remark 2.** Note that in Definitions 2.1 and 2.2,  $\mathbf{v}' \in L^2(0, T; \mathbf{V}_0^*)$  can be easily extended to  $L^2(0, T; \mathbf{V}^*)$  by setting  $\langle \mathbf{v}'(t), \mathbf{1} \rangle_{\mathbf{V}^*, \mathbf{V}} := 0$ , namely

$$\langle \mathbf{v}'(t), \mathbf{z} \rangle_{\mathbf{V}^*, \mathbf{V}} := \langle \mathbf{v}'(t), \mathbf{P}\mathbf{z} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} \quad \text{for all } \mathbf{z} \in \mathbf{V},$$

because we know the following orthogonal decomposition  $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{R}$  with  $\mathbf{R} = \{r\mathbf{1} : r \in \mathbb{R}\}$ . Therefore, we have

$$\langle \mathbf{v}'(t), \mathbf{1} \rangle_{\mathbf{V}^*, \mathbf{V}} = \langle \mathbf{v}'(t), \mathbf{P}\mathbf{1} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} = \langle \mathbf{v}'(t), \mathbf{1} - m(\mathbf{1})\mathbf{1} \rangle_{\mathbf{V}_0^*, \mathbf{V}_0}$$

and it is clear that  $\mathbf{1} - m(\mathbf{1}) = 0$  in  $\mathbf{V}_0$ . Thus, we can identify the dual space  $\mathbf{V}_0^*$  by  $\{\mathbf{z}^* \in \mathbf{V}^* : \langle \mathbf{z}^*, \mathbf{1} \rangle_{\mathbf{V}^*, \mathbf{V}} = 0\}$  and we know that (2.11) holds for all  $\mathbf{z} \in \mathbf{V}$ , that is, its variational formulation implies the structure of volume conservation (see, e.g., [22, p. 5674]).

Now, it is straightforward to check that (2.22) and (2.23) yield the single abstract equation

$$\begin{aligned} \mathbf{F}^{-1}(\mathbf{v}'(t)) + \partial\varphi(\mathbf{v}(t)) &= \mathbf{P}(-\boldsymbol{\xi}(t) - \boldsymbol{\pi}(\mathbf{v}(t) + m_0\mathbf{1}) + \mathbf{f}(t)) \\ &\text{in } \mathbf{H}_0, \text{ for a.a. } t \in (0, T), \end{aligned} \quad (2.28)$$

where

$$\boldsymbol{\xi}(t) \in -\boldsymbol{\beta}(\mathbf{v}(t) + m_0\mathbf{1}) \quad \text{in } \mathbf{H}, \text{ for a.a. } t \in (0, T). \quad (2.29)$$

Based on the abstract theory developed in [13], we expect that (2.28)–(2.29) can be solved by the approach for doubly nonlinear evolution inclusions, which will be discussed in Section 4.

### 3 Continuous dependence

In this section, we prove Theorem 2.1.

**Proof of Theorem 2.1.** For  $i = 1, 2$  let  $(\mathbf{v}^{(i)}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\xi}^{(i)})$  be a weak solution of (P) corresponding to the data  $(\mathbf{f}^{(i)}, \mathbf{v}_0^{(i)})$ . We consider the difference between equations (2.28) written, at the time  $s \in (0, T)$ , for  $\mathbf{v}^{(1)}(s) = (v^{(1)}(s), v_\Gamma^{(1)}(s))$  and  $\mathbf{v}^{(2)}(s) = (v^{(2)}(s), v_\Gamma^{(2)}(s))$ , respectively. Then, we take the inner product with  $\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)$  in  $\mathbf{H}_0$ . Using the property (2.27) as well as (2.29) and the monotonicity of  $\beta, \beta_\Gamma$  we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{ds} |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0^*}^2 + |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0}^2 \\ &\leq \langle \mathbf{f}^{(1)}(s) - \mathbf{f}^{(2)}(s), \mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s) \rangle_{\mathbf{V}^*, \mathbf{V}} \\ &\quad - \left( \pi(v^{(1)}(s) + m_0) - \pi(v^{(2)}(s) + m_0), v^{(1)}(s) - v^{(2)}(s) \right)_H \\ &\quad - \left( \pi_\Gamma(v_\Gamma^{(1)}(s) + m_0) - \pi_\Gamma(v_\Gamma^{(2)}(s) + m_0), v_\Gamma^{(1)}(s) - v_\Gamma^{(2)}(s) \right)_{H_\Gamma} \end{aligned} \quad (3.1)$$

for a.a.  $s \in (0, T)$ . By virtue of the compact embedding of  $\mathbf{V}_0$  into  $\mathbf{H}_0$ , for each  $\delta > 0$  there exists a positive constant  $c_\delta$  (depending on  $\delta$ ) such that

$$|\mathbf{z}|_{\mathbf{H}_0} \leq \delta |\mathbf{z}|_{\mathbf{V}_0} + c_\delta |\mathbf{z}|_{\mathbf{V}_0^*} \quad \text{for all } \mathbf{z} \in \mathbf{V}_0, \quad (3.2)$$

(see, e.g., [23, Sect. 8, Lemma 8]). Then, in view of (2.10) and the Lipschitz continuities of  $\pi$  and  $\pi_\Gamma$  we infer that

$$\begin{aligned} &\frac{1}{2} \frac{d}{ds} |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0^*}^2 + c_p |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}}^2 \\ &\leq \frac{1}{c_p} |\mathbf{f}^{(1)}(s) - \mathbf{f}^{(2)}(s)|_{\mathbf{V}^*}^2 + \frac{c_p}{4} |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}}^2 \\ &\quad + (L + L_\Gamma) \left\{ 2\delta |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0}^2 + 2c_\delta |\mathbf{v}^{(1)}(s) - \mathbf{v}^{(2)}(s)|_{\mathbf{V}_0^*}^2 \right\}, \end{aligned}$$

for a.a.  $s \in (0, T)$ , Therefore, taking  $\delta = c_p/(8(L+L_\Gamma))$  and applying the Gronwall lemma, we find a constant  $C > 0$ , with the dependencies specified in the statement, such that (2.19) holds.  $\square$

## 4 Existence

This section is devoted to the proof of Theorem 2.2. We make use of the Yosida approximation for maximal monotone operators  $\beta$ ,  $\beta_\Gamma$  and of well-known results of this theory (see [2, 3, 20]). For each  $\varepsilon \in (0, 1]$ , we define  $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$ , along with the associated resolvent operators  $J_\varepsilon, J_{\Gamma,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\begin{aligned}\beta_\varepsilon(r) &:= \frac{1}{\varepsilon}(r - J_\varepsilon(r)) := \frac{1}{\varepsilon}(r - (I + \varepsilon\beta)^{-1}(r)), \\ \beta_{\Gamma,\varepsilon}(r) &:= \frac{1}{\varepsilon\varrho}(r - J_{\Gamma,\varepsilon}(r)) := \frac{1}{\varepsilon\varrho}(r - (I + \varepsilon\varrho\beta_\Gamma)^{-1}(r))\end{aligned}$$

for all  $r \in \mathbb{R}$ , where  $\varrho > 0$  is the same constant as in the assumption (2.20). Note that the two definitions are not symmetric since in the second it is  $\varepsilon\varrho$  and not directly  $\varepsilon$  to be used as approximation parameter. Now, we easily have  $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$ . Moreover, the related Moreau–Yosida regularizations  $\widehat{\beta}_\varepsilon, \widehat{\beta}_{\Gamma,\varepsilon}$  of  $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \rightarrow \mathbb{R}$  fulfill

$$\begin{aligned}\widehat{\beta}_\varepsilon(r) &:= \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon}|r - s|^2 + \widehat{\beta}(s) \right\} = \frac{1}{2\varepsilon}|r - J_\varepsilon(r)|^2 + \widehat{\beta}(J_\varepsilon(r)) = \int_0^r \beta_\varepsilon(s) ds, \\ \widehat{\beta}_{\Gamma,\varepsilon}(r) &:= \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon\varrho}|r - s|^2 + \widehat{\beta}_\Gamma(s) \right\} = \int_0^r \beta_{\Gamma,\varepsilon}(s) ds \quad \text{for all } r \in \mathbb{R}.\end{aligned}$$

It is well known that  $\beta_\varepsilon$  is Lipschitz continuous with Lipschitz constant  $1/\varepsilon$  and  $\beta_{\Gamma,\varepsilon}$  is also Lipschitz continuous with constant  $1/(\varepsilon\varrho)$ . In addition, we have the standard properties

$$\begin{aligned}|\beta_\varepsilon(r)| &\leq |\beta^\circ(r)|, \quad |\beta_{\Gamma,\varepsilon}(r)| \leq |\beta_\Gamma^\circ(r)| \quad \text{and} \\ 0 &\leq \widehat{\beta}_\varepsilon(r) \leq \widehat{\beta}(r), \quad 0 \leq \widehat{\beta}_{\Gamma,\varepsilon}(r) \leq \widehat{\beta}_\Gamma(r) \quad \text{for all } r \in \mathbb{R}.\end{aligned}\tag{4.1}$$

Here, thanks to [6, Lemma 4.4], we have that

$$|\beta_\varepsilon(r)| \leq \varrho |\beta_{\Gamma,\varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R},\tag{4.2}$$

with the same constants  $\varrho$  and  $c_0$  as in (2.20).

### 4.1 Approximation of the problem

In this subsection, we consider an approximation for (2.28) stated as the following Cauchy problem: for each  $\varepsilon \in (0, 1]$  find  $\mathbf{v}_\varepsilon := (v_\varepsilon, v_{\Gamma,\varepsilon})$  satisfying

$$\begin{aligned}\varepsilon \mathbf{v}'_\varepsilon(t) + \mathbf{F}^{-1} \mathbf{v}'_\varepsilon(t) + \partial\varphi(\mathbf{v}_\varepsilon(t)) \\ = \mathbf{P}(-\beta_\varepsilon(v_\varepsilon(t) + m_0 \mathbf{1}) - \pi(v_\varepsilon(t) + m_0 \mathbf{1}) + \mathbf{f}(t)) \quad \text{in } \mathbf{H}_0, \text{ for a.a. } t \in (0, T),\end{aligned}\tag{4.3}$$

$$\mathbf{v}_\varepsilon(0) = \mathbf{v}_0 \quad \text{in } \mathbf{H}_0.\tag{4.4}$$

The structure of this approximate problem fits into the framework of the general problem treated in [13]. Namely, thanks to the abstract theory of doubly nonlinear evolution inclusions, we can solve the Cauchy problem (4.3)–(4.4).

**Proposition 4.1.** *For each  $\varepsilon \in (0, 1]$ , there exists a unique*

$$\mathbf{v}_\varepsilon \in H^1(0, T; \mathbf{H}_0) \cap C([0, T]; \mathbf{V}_0) \cap L^2(0, T; \mathbf{W}),$$

*such that  $\mathbf{v}_\varepsilon$  satisfies (4.3)–(4.4).*

**Proof.** As the argumentation follows the analogous proof performed in [11], we only sketch it. We claim that for a given  $\bar{\mathbf{v}} := (\bar{v}, \bar{v}_\Gamma) \in C([0, T]; \mathbf{H}_0)$  there exists a unique

$$\mathbf{v} \in H^1(0, T; \mathbf{H}_0) \cap L^\infty(0, T; \mathbf{V}_0) \subset C([0, T]; \mathbf{H}_0)$$

such that

$$\begin{aligned} & (\varepsilon \mathbf{I} + \mathbf{F}^{-1}) \mathbf{v}'(t) + \partial\varphi(\mathbf{v}(t)) \\ & \ni \mathbf{P}(-\beta_\varepsilon(\bar{\mathbf{v}}(t) + m_0 \mathbf{1}) - \pi(\bar{\mathbf{v}}(t) + m_0 \mathbf{1}) + \mathbf{f}(t)) \quad \text{in } \mathbf{H}_0, \text{ for a.a. } t \in (0, T), \\ & \mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \mathbf{H}_0. \end{aligned}$$

Indeed, it suffices to apply the abstract theory of doubly nonlinear evolution inclusions (see in particular [13, Thm. 2.1]). We point out that, thanks to the presence of  $\varepsilon \mathbf{I}$ , the operator  $\varepsilon \mathbf{I} + \mathbf{F}^{-1}$  is coercive in  $\mathbf{H}_0$ , which is an important assumption of Theorem 2.1 in [13]. Then, we construct the map  $\Psi : \bar{\mathbf{v}} \mapsto \mathbf{v}$  from  $C([0, T]; \mathbf{H}_0)$  into itself. Next, for a given  $\bar{\mathbf{v}}^{(i)} \in C([0, T]; \mathbf{H}_0)$ , we put  $\mathbf{v}^{(i)} := \Psi \bar{\mathbf{v}}^{(i)}$ ,  $i = 1, 2$ . Using the monotonicity of  $\partial\varphi$ , it is not difficult to deduce the estimate

$$|\mathbf{v}^{(1)}(t) - \mathbf{v}^{(2)}(t)|_{\mathbf{H}_0}^2 \leq c_\varepsilon \int_0^t |\bar{\mathbf{v}}^{(1)}(s) - \bar{\mathbf{v}}^{(2)}(s)|_{\mathbf{H}_0}^2 ds \quad \text{for all } t \in [0, T],$$

where  $c_\varepsilon$  is a constant depending on  $L$ ,  $L_\Gamma$  and  $\varepsilon$ . Therefore, we can prove that there exists a suitable  $k \in \mathbb{N}$  such that  $\Psi^k$  is a contraction mapping in  $C([0, T]; \mathbf{H}_0)$ . Hence, being  $\varepsilon > 0$  there exists a unique fixed point for  $\Psi$  which yields the unique solution  $\mathbf{v}_\varepsilon$  of the problem (4.3)–(4.4). Finally, thanks to the fact that  $\partial\varphi(\mathbf{v}_\varepsilon) \in L^2(0, T; \mathbf{H}_0)$ , we easily reach the conclusion, by simply observing that  $H^1(0, T; \mathbf{H}_0) \cap L^2(0, T; \mathbf{W})$  is contained in  $C([0, T]; \mathbf{V}_0)$ .  $\square$

Now, for each  $\varepsilon \in (0, 1]$  we set

$$\boldsymbol{\mu}_\varepsilon(t) := \varepsilon \mathbf{v}'_\varepsilon(t) + \partial\varphi(\mathbf{v}_\varepsilon(t)) + \beta_\varepsilon(\mathbf{u}_\varepsilon(t)) + \pi(\mathbf{u}_\varepsilon(t)) - \mathbf{f}(t) \quad \text{for a.a. } t \in (0, T), \quad (4.5)$$

where  $\mathbf{u}_\varepsilon := \mathbf{v}_\varepsilon + m_0 \mathbf{1} = (v_\varepsilon + m_0, v_{\Gamma, \varepsilon} + m_0)$ . We expect (4.5) to give the approximate sequence for the chemical potential. Then, we can rewrite the evolution equation (4.3) as

$$\mathbf{F}^{-1}(\mathbf{v}'_\varepsilon(t)) + \boldsymbol{\mu}_\varepsilon(t) - \omega_\varepsilon(t) \mathbf{1} = \mathbf{0} \quad \text{in } \mathbf{V}, \text{ for a.a. } t \in (0, T),$$

where

$$\omega_\varepsilon(t) := m(\beta_\varepsilon(\mathbf{u}_\varepsilon(t)) + \pi(\mathbf{u}_\varepsilon(t)) - \mathbf{f}(t)), \quad t \in (0, T). \quad (4.6)$$

Hence, we realize that  $\mathbf{P}\boldsymbol{\mu}_\varepsilon = \boldsymbol{\mu}_\varepsilon - \omega_\varepsilon \mathbf{1} \in L^2(0, T; \mathbf{V}_0)$  and  $\omega_\varepsilon \in L^2(0, T)$ . From these regularities it follows that  $\boldsymbol{\mu}_\varepsilon \in L^2(0, T; \mathbf{V})$  and the pair  $(\mathbf{v}_\varepsilon, \boldsymbol{\mu}_\varepsilon)$  satisfies

$$\mathbf{v}'_\varepsilon(t) + \mathbf{F}(\mathbf{P}\boldsymbol{\mu}_\varepsilon(t)) = \mathbf{0} \quad \text{in } \mathbf{V}_0^*, \text{ for a.a. } t \in (0, T). \quad (4.7)$$

Moreover, we see that  $(v_\varepsilon, v_{\Gamma,\varepsilon}, \mu_\varepsilon, \mu_{\Gamma,\varepsilon})$  fulfill the following weak formulations:

$$\begin{aligned} \int_{\Omega} \partial_t v_\varepsilon(t) z dx + \int_{\Gamma} \partial_t v_{\Gamma,\varepsilon}(t) z_\Gamma d\Gamma + \int_{\Omega} \nabla \mu_\varepsilon(t) \cdot \nabla z dx \\ + \int_{\Gamma} \nabla_{\Gamma} \mu_{\Gamma,\varepsilon}(t) \cdot \nabla_{\Gamma} z_\Gamma d\Gamma = 0 \quad \text{for all } \mathbf{z} = (z, z_\Gamma) \in \mathbf{V}_0, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \int_{\Omega} \mu_\varepsilon(t) z dx + \int_{\Gamma} \mu_{\Gamma,\varepsilon}(t) z_\Gamma d\Gamma \\ = \varepsilon \int_{\Omega} \partial_t v_\varepsilon(t) z dx + \varepsilon \int_{\Gamma} \partial_t v_{\Gamma,\varepsilon}(t) z_\Gamma d\Gamma + \int_{\Omega} \nabla v_\varepsilon(t) \cdot \nabla z dx + \int_{\Gamma} \nabla_{\Gamma} v_{\Gamma,\varepsilon}(t) \cdot \nabla_{\Gamma} z_\Gamma d\Gamma \\ + \int_{\Omega} (\beta_\varepsilon(v_\varepsilon(t) + m_0) + \pi(v_\varepsilon(t) + m_0) - f(t)) z dx \\ + \int_{\Gamma} (\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t) + m_0) + \pi_{\Gamma}(v_{\Gamma,\varepsilon}(t) + m_0) - f_{\Gamma}(t)) z_\Gamma d\Gamma \\ \text{for all } \mathbf{z} = (z, z_\Gamma) \in \mathbf{V}, \end{aligned} \quad (4.9)$$

for a.a.  $t \in (0, T)$ . In particular, from (4.9) we deduce the equations

$$\mu_\varepsilon = \varepsilon \partial_t v_\varepsilon - \Delta v_\varepsilon + \beta_\varepsilon(v_\varepsilon + m_0) + \pi(v_\varepsilon + m_0) - f \quad \text{a.e. in } Q, \quad (4.10)$$

$$\mu_{\Gamma,\varepsilon} = \varepsilon \partial_t v_{\Gamma,\varepsilon} + \partial_\nu v_\varepsilon - \Delta_{\Gamma} v_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon} + m_0) + \pi_{\Gamma}(v_{\Gamma,\varepsilon} + m_0) - f_{\Gamma} \quad \text{a.e. on } \Sigma. \quad (4.11)$$

Arguing on (4.8), since  $\partial_t v_\varepsilon \in L^2(0, T; H)$  and  $\partial_t v_{\Gamma,\varepsilon} \in L^2(0, T; H_{\Gamma})$ , we can recover that  $\Delta \mu_\varepsilon \in L^2(0, T; H)$ . On the other hand, we already know that  $\mu_\varepsilon \in L^2(0, T; V)$  and  $\mu_{\Gamma,\varepsilon} \in L^2(0, T; V_{\Gamma})$ . Then, we infer that (see, e.g., [4, Thm. 3.2, p. 1.79])

$$\mu_\varepsilon \in L^2(0, T; H^{3/2}(\Omega))$$

and consequently, by a trace theorem [4, Thm. 2.27, p. 1.64],  $\partial_\nu \mu_\varepsilon \in L^2(0, T; H_{\Gamma})$ . Therefore, we also obtain that  $\Delta_{\Gamma} \mu_{\Gamma,\varepsilon} \in L^2(0, T; H_{\Gamma})$  so that we can write the equations

$$\partial_t v_\varepsilon - \Delta \mu_\varepsilon = 0 \quad \text{a.e. in } Q, \quad (4.12)$$

$$\partial_t v_{\Gamma,\varepsilon} + \partial_\nu \mu_\varepsilon - \Delta_{\Gamma} \mu_{\Gamma,\varepsilon} = 0 \quad \text{a.e. on } \Sigma. \quad (4.13)$$

Moreover, the additional information  $\Delta_{\Gamma} \mu_{\Gamma,\varepsilon} \in L^2(0, T; H_{\Gamma})$  implies (see, e.g., [19, p. 104])  $\mu_{\Gamma,\varepsilon} \in L^2(0, T; H^2(\Gamma))$ . Finally, this yields in particular that  $\mu_{\Gamma,\varepsilon} \in L^2(0, T; H^{3/2}(\Gamma))$ , whence (quoting again [4, Thm. 3.2, p. 1.79]) we obtain  $\mu_\varepsilon \in L^2(0, T; H^2(\Omega))$ , that is

$$\boldsymbol{\mu}_\varepsilon \in L^2(0, T; \mathbf{W}).$$

From the next subsection, we can proceed with the a priori estimates.

## 4.2 A priori estimates

In this subsection, we obtain the uniform estimates independent of  $\varepsilon$ . We can adopt the same strategy as in [6, 10–12]. Here, we use systematically the relations

$$\mathbf{u}_\varepsilon = (u_\varepsilon, u_{\Gamma,\varepsilon}) = \mathbf{v}_\varepsilon + m_0 \mathbf{1} = (v_\varepsilon + m_0, v_{\Gamma,\varepsilon} + m_0).$$

**Lemma 4.1.** *There exists a positive constant  $M_1$ , independent of  $\varepsilon \in (0, 1]$ , such that*

$$\begin{aligned} & \varepsilon^{1/2} |\mathbf{v}_\varepsilon|_{L^\infty(0,T;\mathbf{H}_0)} + |\mathbf{v}_\varepsilon|_{L^\infty(0,T;\mathbf{V}_0^*)} + |\mathbf{v}_\varepsilon|_{L^2(0,T;\mathbf{V}_0)} \\ & + |\beta_\varepsilon(u_\varepsilon)|_{L^1(0,T;L^1(\Omega))} + |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^1(0,T;L^1(\Gamma))} \leq M_1. \end{aligned} \quad (4.14)$$

**Proof.** We test (4.3) at time  $s \in (0, T)$  by  $\mathbf{v}_\varepsilon(s) \in \mathbf{V}_0$ . Then from (2.10), (2.27) and the definition of the subdifferential with  $\varphi(\mathbf{0}) = 0$  we see that

$$\begin{aligned} & (\varepsilon \mathbf{v}'_\varepsilon(s), \mathbf{v}_\varepsilon(s))_{\mathbf{H}_0} + (\mathbf{v}'_\varepsilon(s), \mathbf{v}_\varepsilon(s))_{\mathbf{V}_0^*} + \varphi(\mathbf{v}_\varepsilon(s)) \\ & + (\beta_\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} \leq (\mathbf{f}(s) - \boldsymbol{\pi}(\mathbf{u}_\varepsilon(s)), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} \quad \text{for a.a. } s \in (0, T). \end{aligned} \quad (4.15)$$

Now, let us recall the assumptions  $D(\beta_\Gamma) \subseteq D(\beta)$  of (A6) and  $m_0 \in \text{int} D(\beta_\Gamma)$  of (A7). Hence, we can exploit the useful inequalities (a proof is given in [16, Sect. 5]) stating the existence of two constants  $\delta_0 > 0$  and  $c_1 > 0$  such that

$$\beta_\varepsilon(r)(r - m_0) \geq \delta_0 |\beta_\varepsilon(r)| - c_1, \quad \beta_{\Gamma,\varepsilon}(r)(r - m_0) \geq \delta_0 |\beta_{\Gamma,\varepsilon}(r)| - c_1$$

for all  $r \in \mathbb{R}$  and  $\varepsilon \in (0, 1]$ . Therefore, we easily have

$$\begin{aligned} & (\beta_\varepsilon(\mathbf{u}_\varepsilon(s)), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} \\ & = \int_\Omega \beta_\varepsilon(u_\varepsilon(s))(u_\varepsilon(s) - m_0) dx + \int_\Gamma \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))(u_{\Gamma,\varepsilon}(s) - m_0) d\Gamma \\ & \geq \delta_0 \int_\Omega |\beta_\varepsilon(u_\varepsilon(s))| dx - c_1 |\Omega| + \delta_0 \int_\Gamma |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| d\Gamma - c_1 |\Gamma| \end{aligned} \quad (4.16)$$

for a.a.  $s \in (0, T)$ . Moreover, in view of the assumption (A4) and the compactness inequality (3.2), there exists a positive constant  $\tilde{M}_1$ , depending only on  $L$ ,  $L_\Gamma$ ,  $\pi(m_0)$ ,  $\pi_\Gamma(m_0)$ ,  $|\Omega|$  and  $|\Gamma|$ , such that

$$\begin{aligned} & (\mathbf{f}(s) - \boldsymbol{\pi}(\mathbf{u}_\varepsilon(s)), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} \\ & \leq \frac{1}{2} \int_\Omega |f(s)|^2 dx + \frac{1}{2} \int_\Omega |v_\varepsilon(s)|^2 dx + L \int_\Omega |v_\varepsilon(s)|^2 dx + |\pi(m_0)| \int_\Omega |v_\varepsilon(s)| dx \\ & \quad + \frac{1}{2} \int_\Gamma |f_\Gamma(s)|^2 d\Gamma + \frac{1}{2} \int_\Gamma |v_{\Gamma,\varepsilon}(s)|^2 d\Gamma + L_\Gamma \int_\Gamma |v_{\Gamma,\varepsilon}(s)|^2 d\Gamma + |\pi_\Gamma(m_0)| \int_\Gamma |v_{\Gamma,\varepsilon}(s)| d\Gamma \\ & \leq \tilde{M}_1 \left( 1 + |\mathbf{f}(s)|_{\mathbf{H}}^2 + |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 \right) + \frac{1}{4} |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0}^2 \end{aligned} \quad (4.17)$$

for a.a.  $s \in (0, T)$ . Therefore, collecting (4.15)–(4.17) it is straightforward to obtain

$$\begin{aligned} & \varepsilon \frac{d}{ds} |\mathbf{v}_\varepsilon(s)|_{\mathbf{H}_0}^2 + \frac{d}{ds} |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 + \frac{1}{2} |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0}^2 \\ & + 2\delta_0 \int_\Omega |\beta_\varepsilon(u_\varepsilon(s))| dx + 2\delta_0 \int_\Gamma |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| d\Gamma \\ & \leq 2c_1 (|\Omega| + |\Gamma|) + 2\tilde{M}_1 \left( 1 + |\mathbf{f}(s)|_{\mathbf{H}}^2 + |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 \right) \quad \text{for a.a. } s \in (0, T). \end{aligned}$$

Thus, the Gronwall inequality leads us to the conclusion.  $\square$

Using the regularity assumption (A5) for  $\mathbf{f}$ , we can show the following estimate.

**Lemma 4.2.** *There exists a positive constant  $M_2$ , independent of  $\varepsilon \in (0, 1]$ , such that*

$$\begin{aligned} & \varepsilon^{1/2} |\mathbf{v}'_\varepsilon|_{L^2(0,T;\mathbf{H}_0)} + |\mathbf{v}'_\varepsilon|_{L^2(0,T;\mathbf{V}_0^*)} + |\mathbf{v}_\varepsilon|_{L^\infty(0,T;\mathbf{V}_0)} \\ & + |\widehat{\beta}_\varepsilon(u_\varepsilon)|_{L^\infty(0,T;L^1(\Omega))} + |\widehat{\beta}_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^\infty(0,T;L^1(\Gamma))} \leq M_2. \end{aligned} \quad (4.18)$$

**Proof.** We test (4.3) at time  $s \in (0, T)$  by  $\mathbf{v}'_\varepsilon(s) \in \mathbf{H}_0$ . Then, with the help of (2.27) we have

$$\begin{aligned} & \varepsilon |\mathbf{v}'_\varepsilon(s)|_{\mathbf{H}_0}^2 + |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}^2 + \frac{d}{ds} \varphi(\mathbf{v}_\varepsilon(s)) \\ & + \frac{d}{ds} \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0) dx + \frac{d}{ds} \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) d\Gamma \\ & \leq \frac{1}{4} |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}^2 + \int_\Omega |\nabla \pi(v_\varepsilon(s) + m_0)|^2 dx + \int_\Gamma |\nabla_\Gamma \pi_\Gamma(v_{\Gamma,\varepsilon}(s) + m_0)|^2 d\Gamma + (\mathbf{f}(s), \mathbf{v}'_\varepsilon(s))_{\mathbf{H}} \end{aligned}$$

for a.a.  $s \in (0, T)$ . Integrating it over  $(0, t)$  with respect to  $s$ , in view of (4.1), (A2) and (A7) we infer that

$$\begin{aligned} & \varepsilon \int_0^t |\mathbf{v}'_\varepsilon(s)|_{\mathbf{H}_0}^2 ds + \frac{3}{4} \int_0^t |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}^2 ds + \frac{1}{2} |\mathbf{v}_\varepsilon(t)|_{\mathbf{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon(t)) dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t)) d\Gamma \\ & \leq \frac{1}{2} |\mathbf{v}_0|_{\mathbf{V}_0}^2 + \int_\Omega \widehat{\beta}(u_0) dx + \int_\Gamma \widehat{\beta}_\Gamma(u_{0\Gamma}) d\Gamma \\ & + L^2 \int_0^t \int_\Omega |\nabla v_\varepsilon(s)|^2 dx ds + L_\Gamma^2 \int_0^t \int_\Gamma |\nabla_\Gamma v_\varepsilon(s)|^2 d\Gamma ds + \int_0^t (\mathbf{f}(s), \mathbf{v}'_\varepsilon(s))_{\mathbf{H}} ds \end{aligned} \quad (4.19)$$

for all  $t \in [0, T]$ . Let us now recall (A5): if  $\mathbf{f} \in W^{1,1}(0, T; \mathbf{H})$ , we can integrate the last term of (4.19) by parts in time. With the help of the Young inequality and (2.10) we easily obtain

$$\begin{aligned} & \int_0^t (\mathbf{f}(s), \mathbf{v}'_\varepsilon(s))_{\mathbf{H}} ds \\ & = (\mathbf{f}(t), \mathbf{v}_\varepsilon(t))_{\mathbf{H}} - (\mathbf{f}(0), \mathbf{v}_0)_{\mathbf{H}} - \int_0^t (\mathbf{f}'(s), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} ds \\ & \leq \frac{1}{4} |\mathbf{v}_\varepsilon(t)|_{\mathbf{V}_0}^2 + \frac{1}{4} |\mathbf{v}_0|_{\mathbf{H}}^2 + \left( \frac{1}{c_p} + 1 \right) |\mathbf{f}|_{C([0,T];\mathbf{H})}^2 + \frac{1}{c_p^{1/2}} \int_0^t |\mathbf{f}'(s)|_{\mathbf{H}} |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0} ds. \end{aligned}$$

Combinig this with (4.19) and applying the Gronwall lemma in the form of [3, Lemme A.5, p. 157], there is a positive constant  $M_2$ , depending only on  $|\mathbf{v}_0|_{\mathbf{V}_0}$ ,  $c_p$ ,  $|\widehat{\beta}(u_0)|_{L^1(\Omega)}$ ,  $|\widehat{\beta}_\Gamma(u_{0\Gamma})|_{L^1(\Gamma)}$ ,  $L$ ,  $L_\Gamma$ ,  $M_1$  and  $|\mathbf{f}|_{W^{1,1}(0,T;\mathbf{H})}$ , such that (4.18) holds. On the other hand, if  $\mathbf{f} \in L^2(0, T; \mathbf{V})$  in (A5) it suffices to observe that

$$\int_0^t (\mathbf{f}(s), \mathbf{v}'_\varepsilon(s))_{\mathbf{H}} ds = \int_0^t \langle \mathbf{v}'_\varepsilon(s), \mathbf{P}\mathbf{f}(s) \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} ds \leq \frac{1}{4} \int_0^t |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}^2 ds + |\mathbf{f}|_{L^2(0,T;\mathbf{V})}^2$$

and collecting this and (4.19) leads to the estimate (4.18) with a small change in the dependencies of  $M_2$ .  $\square$

**Lemma 4.3.** *There exist two positive constants  $M_3$  and  $M_4$ , independent of  $\varepsilon \in (0, 1]$ , such that*

$$|\beta_\varepsilon(u_\varepsilon)|_{L^2(0,T;L^1(\Omega))} + |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^2(0,T;L^1(\Gamma))} \leq M_3, \quad (4.20)$$

$$|\omega_\varepsilon|_{L^2(0,T)} + |\boldsymbol{\mu}_\varepsilon|_{L^2(0,T;\mathbf{V})} \leq M_4. \quad (4.21)$$

**Proof.** Recalling (4.15) and (4.16), we easily infer that

$$\begin{aligned} & \delta_0 \int_{\Omega} |\beta_\varepsilon(u_\varepsilon(s))| dx + \delta_0 \int_{\Gamma} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| d\Gamma \\ & \leq c_1(|\Omega| + |\Gamma|) + (\mathbf{f}(s) - \boldsymbol{\pi}(\mathbf{u}_\varepsilon(s)) - \varepsilon \mathbf{v}'_\varepsilon(s), \mathbf{v}_\varepsilon(s))_{\mathbf{H}} - (\mathbf{v}'_\varepsilon(s), \mathbf{v}_\varepsilon(s))_{\mathbf{V}_0^*} \end{aligned} \quad (4.22)$$

for a.a.  $s \in (0, T)$ . Hence, by squaring we have

$$\begin{aligned} & \left( \delta_0 \int_{\Omega} |\beta_\varepsilon(u_\varepsilon(s))| dx + \delta_0 \int_{\Gamma} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| d\Gamma \right)^2 \\ & \leq 3c_1^2(|\Omega| + |\Gamma|)^2 + 9 \left( |\mathbf{f}(s)|_{\mathbf{H}}^2 + |\boldsymbol{\pi}(\mathbf{u}_\varepsilon(s))|_{\mathbf{H}}^2 + \varepsilon^2 |\mathbf{v}'_\varepsilon(s)|_{\mathbf{H}_0}^2 \right) |\mathbf{v}_\varepsilon(s)|_{\mathbf{H}_0}^2 \\ & \quad + 3 |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}^2 |\mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 \quad \text{for a.a. } s \in (0, T). \end{aligned}$$

Then, due to Lemma 4.2, there exists a positive constant  $M_3$ , depending on  $\delta_0$ ,  $c_1$ ,  $T$ ,  $|\Omega|$ ,  $|\Gamma|$ ,  $M_2$ ,  $L$ ,  $L_\Gamma$ ,  $|\pi(m_0)|$ ,  $|\pi_\Gamma(m_0)|$ ,  $|\mathbf{f}|_{L^2(0,T;\mathbf{H})}$  and independent of  $\varepsilon \in (0, 1]$ , such that (4.20) holds. Next, from the definition of  $\omega_\varepsilon$ , given by (4.6), we have

$$\begin{aligned} |\omega_\varepsilon(t)|^2 & \leq \frac{6}{(|\Omega| + |\Gamma|)^2} \left\{ |\beta_\varepsilon(u_\varepsilon(t))|_{L^1(\Omega)}^2 + |\pi(u_\varepsilon(t))|_{L^1(\Omega)}^2 + |\Omega| |f(t)|_{\mathbf{H}}^2 \right. \\ & \quad \left. + |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(t))|_{L^1(\Gamma)}^2 + |\pi_\Gamma(u_{\Gamma,\varepsilon}(t))|_{L^1(\Gamma)}^2 + |\Gamma| |f_\Gamma(t)|_{\mathbf{H}_\Gamma}^2 \right\}. \end{aligned}$$

Then, by integrating over  $(0, T)$ , it follows that there is a positive constant  $\tilde{M}_4$ , depending only on  $|\Omega|$ ,  $|\Gamma|$ ,  $M_2$ ,  $M_3$ ,  $L$ ,  $L_\Gamma$ ,  $|\pi(m_0)|$ ,  $|\pi_\Gamma(m_0)|$  and  $|\mathbf{f}|_{L^2(0,T;\mathbf{H})}$ , such that

$$|\omega_\varepsilon|_{L^2(0,T)} \leq \tilde{M}_4.$$

At this point, we test (4.7) at time  $s \in (0, T)$  by  $\mathbf{P}\boldsymbol{\mu}_\varepsilon(s) \in \mathbf{V}_0$  and obtain

$$\langle \mathbf{F}(\mathbf{P}\boldsymbol{\mu}_\varepsilon(s)), \mathbf{P}\boldsymbol{\mu}_\varepsilon(s) \rangle_{\mathbf{V}_0^*, \mathbf{V}_0} = -\langle \mathbf{v}'_\varepsilon(s), \mathbf{P}\boldsymbol{\mu}_\varepsilon(s) \rangle_{\mathbf{V}_0^*, \mathbf{V}_0}$$

for a.a.  $s \in (0, T)$ . Then, in view of (2.10) we deduce that

$$|\mathbf{P}\boldsymbol{\mu}_\varepsilon(s)|_{\mathbf{V}_0} \leq |\mathbf{v}'_\varepsilon(s)|_{\mathbf{V}_0^*}$$

and

$$\begin{aligned} |\boldsymbol{\mu}_\varepsilon(s)|_{\mathbf{V}} & \leq |\mathbf{P}\boldsymbol{\mu}_\varepsilon(s)|_{\mathbf{V}} + |\omega_\varepsilon(s)\mathbf{1}|_{\mathbf{V}} \\ & \leq c_p^{-1/2} |\mathbf{P}\boldsymbol{\mu}_\varepsilon(s)|_{\mathbf{V}_0} + (|\Omega| + |\Gamma|)^{1/2} |\omega_\varepsilon(s)| \end{aligned}$$

whence (4.21) follows by squaring and integrating over  $(0, T)$ , on account of the estimate for  $|\mathbf{v}'_\varepsilon|_{L^2(0,T;\mathbf{V}_0^*)}$  in (4.18).  $\square$



**Lemma 4.4.** *There exist two positive constants  $M_5$  and  $M_6$ , independent of  $\varepsilon \in (0, 1]$ , such that*

$$|\beta_\varepsilon(u_\varepsilon)|_{L^2(0,T;H)} + |\beta_\varepsilon(u_{\Gamma,\varepsilon})|_{L^2(0,T;H_\Gamma)} \leq M_5, \quad (4.23)$$

$$|\Delta v_\varepsilon|_{L^2(0,T;H)} + |v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} + |\partial_\nu v_\varepsilon|_{L^2(0,T;H_\Gamma)} \leq M_6. \quad (4.24)$$

**Proof.** We test (4.10) by  $\beta_\varepsilon(u_\varepsilon) \in L^2(0, T; V)$  and use (4.11), by noting that  $(\beta_\varepsilon(u_\varepsilon))|_\Gamma = \beta_\varepsilon(u_{\Gamma,\varepsilon}) \in L^2(0, T; V_\Gamma)$  as well. Then, by integrating over  $\Omega$ , we infer that

$$\begin{aligned} & \int_\Omega \beta'_\varepsilon(u_\varepsilon(s)) |\nabla v_\varepsilon(s)|^2 dx + |\beta_\varepsilon(u_\varepsilon(s))|_H^2 \\ & + \int_\Gamma \beta'_\varepsilon(u_{\Gamma,\varepsilon}(s)) |\nabla_\Gamma v_{\Gamma,\varepsilon}(s)|^2 d\Gamma + \int_\Gamma \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s)) \beta_\varepsilon(u_{\Gamma,\varepsilon}(s)) d\Gamma \\ & \leq (f(s) + \mu_\varepsilon(s) - \varepsilon \partial_t v_\varepsilon(s) - \pi(u_\varepsilon(s)), \beta_\varepsilon(u_\varepsilon(s)))_H \\ & \quad + (f_\Gamma(s) + \mu_{\Gamma,\varepsilon}(s) - \varepsilon \partial_t v_{\Gamma,\varepsilon}(s) - \pi_\Gamma(u_{\Gamma,\varepsilon}(s)), \beta_\varepsilon(u_{\Gamma,\varepsilon}(s)))_{H_\Gamma} \end{aligned}$$

for a.a.  $s \in (0, T)$ . Recalling now the condition (4.2) we deduce that

$$\begin{aligned} & \int_\Gamma \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s)) \beta_\varepsilon(u_{\Gamma,\varepsilon}(s)) d\Gamma = \int_\Gamma |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))| |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))| d\Gamma \\ & \geq \frac{1}{\varrho} \int_\Gamma |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))|^2 d\Gamma - \frac{c_0}{\varrho} \int_\Gamma |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))| d\Gamma \\ & \geq \frac{1}{2\varrho} \int_\Gamma |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))|^2 d\Gamma - \frac{c_0^2}{2\varrho} |\Gamma|, \end{aligned}$$

because  $\beta_\varepsilon(r)$  and  $\beta_{\Gamma,\varepsilon}(r)$  have the same sign for all  $r \in \mathbb{R}$ . Also, we observe that

$$\int_\Omega \beta'_\varepsilon(u_\varepsilon(s)) |\nabla v_\varepsilon(s)|^2 dx \geq 0, \quad \int_\Gamma \beta'_\varepsilon(u_{\Gamma,\varepsilon}(s)) |\nabla_\Gamma v_{\Gamma,\varepsilon}(s)|^2 d\Gamma \geq 0.$$

Moreover, by the Young inequality and the Lipschitz continuity of  $\pi$  and  $\pi_\Gamma$  we can find a positive constant  $\tilde{M}_5$ , independent of  $\varepsilon \in (0, 1]$ , such that

$$\begin{aligned} & (f(s) + \mu_\varepsilon(s) - \pi(u_\varepsilon(s)), \beta_\varepsilon(u_\varepsilon(s)))_H \\ & \leq \frac{1}{2} |\beta_\varepsilon(u_\varepsilon(s))|_H^2 + \tilde{M}_5 (1 + |f(s)|_H^2 + |\mu_\varepsilon(s)|_H^2 + |v_\varepsilon(s)|_H^2 + \varepsilon |\partial_t v_\varepsilon(s)|_H^2), \\ & (f_\Gamma(s) + \mu_{\Gamma,\varepsilon}(s) - \pi_\Gamma(u_{\Gamma,\varepsilon}(s)), \beta_\varepsilon(u_{\Gamma,\varepsilon}(s)))_{H_\Gamma} \\ & \leq \frac{1}{4\varrho} |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))|_{H_\Gamma}^2 + \varrho \tilde{M}_5 (1 + |f_\Gamma(s)|_{H_\Gamma}^2 + |\mu_{\Gamma,\varepsilon}(s)|_{H_\Gamma}^2 + |v_{\Gamma,\varepsilon}(s)|_{H_\Gamma}^2 + \varepsilon |\partial_t v_{\Gamma,\varepsilon}(s)|_{H_\Gamma}^2). \end{aligned}$$

Combining the above inequalities, integrating in  $(0, T)$  with respect to  $s$  and recalling (4.18) and (4.21) we infer that

$$\frac{1}{2} \int_0^T |\beta_\varepsilon(u_\varepsilon(s))|_H^2 ds + \frac{1}{4\varrho} \int_0^T |\beta_\varepsilon(u_{\Gamma,\varepsilon}(s))|_{H_\Gamma}^2 ds$$

is bounded independently of  $\varepsilon$ , that is, there is a positive constant  $M_5$ , independent of  $\varepsilon \in (0, 1]$ , such that

$$|\beta_\varepsilon(u_\varepsilon)|_{L^2(0,T;H)} + |\beta_\varepsilon(u_{\Gamma,\varepsilon})|_{L^2(0,T;H_\Gamma)} \leq M_5.$$

Now, we can compare the terms in (4.10) and conclude that  $|\Delta v_\varepsilon|_{L^2(0,T;H)}$  is uniformly bounded as well. Hence, with the help of Lemma 4.1 and applying the theory of the elliptic regularity (see, e.g., [4, Thm. 3.2, p. 1.79]), we have that

$$|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} \leq \tilde{M}_6$$

and, owing to the trace theory (see, e.g., [4, Thm. 2.25, p. 1.62]), that

$$|\partial_\nu v_\varepsilon|_{L^2(0,T;H_\Gamma)} \leq \tilde{M}_6$$

for some positive constant  $\tilde{M}_6$  that is independent of  $\varepsilon \in (0, 1]$ . Thus, the lemma is completely proved.  $\square$

**Lemma 4.5.** *There exist two positive constants  $M_7$  and  $M_8$ , independent of  $\varepsilon \in (0, 1]$ , such that*

$$|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^2(0,T;H_\Gamma)} \leq M_7, \quad |\mathbf{v}_\varepsilon|_{L^2(0,T;\mathbf{W})} \leq M_8. \quad (4.25)$$

**Proof.** We test (4.11) by  $\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}) \in L^2(0,T;V_\Gamma)$  and integrate it on the boundary  $\Gamma$ , deducing that

$$\begin{aligned} & \int_\Gamma \beta'_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s)) |\nabla_\Gamma v_{\Gamma,\varepsilon}(s)|^2 d\Gamma + |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))|_{H_\Gamma}^2 \\ & \leq (f_\Gamma(s) + \mu_\Gamma(s) - \varepsilon \partial_t v_{\Gamma,\varepsilon}(s) - \partial_\nu v_\varepsilon(s) - \pi_\Gamma(u_{\Gamma,\varepsilon}(s)), \beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s)))_{H_\Gamma} \\ & \leq \frac{1}{2} |\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))|_{H_\Gamma}^2 \\ & \quad + 3 \left( |f_\Gamma(s)|_{H_\Gamma}^2 + |\mu_\Gamma(s)|_{H_\Gamma}^2 + \varepsilon |\partial_t v_{\Gamma,\varepsilon}(s)|_{H_\Gamma}^2 + |\partial_\nu v_\varepsilon(s)|_{H_\Gamma}^2 + |\pi_\Gamma(u_{\Gamma,\varepsilon}(s))|_{H_\Gamma}^2 \right) \end{aligned}$$

for a.a.  $s \in (0, T)$ . By neglecting the first positive contribution and integrating over  $(0, T)$ , we find out that there is a positive constant  $M_7$ , depending only on  $|f_\Gamma|_{L^2(0,T;H_\Gamma)}$ ,  $M_4$ ,  $M_2$ ,  $M_6$ ,  $L_\Gamma$ ,  $|\pi_\Gamma(m_0)|$ ,  $|\Gamma|$  and  $T$ , such that

$$|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon})|_{L^2(0,T;H_\Gamma)} \leq M_7.$$

Thanks to (4.14) and (4.24), by comparison in (4.11) we also infer that  $|\Delta_\Gamma v_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma)}$  is bounded independently of  $\varepsilon \in (0, 1]$ , and consequently (see, e.g., [19, Sect. 4.2]),

$$|v_{\Gamma,\varepsilon}|_{L^2(0,T;H^2(\Gamma))} \leq \left( |v_{\Gamma,\varepsilon}|_{L^2(0,T;V_\Gamma)}^2 + |\Delta_\Gamma v_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma)}^2 \right)^{1/2} \leq \tilde{M}_8. \quad (4.26)$$

for some constant  $\tilde{M}_8$ . Then, using the theory of the elliptic regularity (see, e.g., [4, Thm. 3.2, p. 1.79]), by virtue of (4.14) and (4.24) it follows that  $|v_\varepsilon|_{L^2(0,T;H^2(\Omega))}$  is uniformly bounded, whence (cf. (4.26))

$$|\mathbf{v}_\varepsilon|_{L^2(0,T;\mathbf{W})} \leq M_8$$

for some positive constant  $M_8$  independent of  $\varepsilon \in (0, 1]$ . Hence, (4.25) is proved.  $\square$

### 4.3 Passage to the limit as $\varepsilon \rightarrow 0$

In this subsection, we conclude the existence proof by passing to the limit in the approximating problem as  $\varepsilon \rightarrow 0$ . Indeed, owing to the estimates stated in Lemmas from 4.1 to 4.5, there exist a subsequence of  $\varepsilon$  (not relabeled) and some limit functions  $\mathbf{v}$ ,  $\boldsymbol{\mu}$ ,  $\xi$ ,  $\xi_\Gamma$  and  $\omega$  such that

$$\mathbf{v}_\varepsilon \rightarrow \mathbf{v} \quad \text{weakly star in } H^1(0, T; \mathbf{V}_0^*) \cap L^\infty(0, T; \mathbf{V}_0) \cap L^2(0, T; \mathbf{W}), \quad (4.27)$$

$$\varepsilon \mathbf{v}_\varepsilon \rightarrow \mathbf{0} \quad \text{strongly in } H^1(0, T; \mathbf{H}_0), \quad (4.28)$$

$$\boldsymbol{\mu}_\varepsilon \rightarrow \boldsymbol{\mu} \quad \text{weakly in } L^2(0, T; \mathbf{V}), \quad (4.29)$$

$$\beta_\varepsilon(u_\varepsilon) \rightarrow \xi \quad \text{weakly in } L^2(0, T; H), \quad (4.30)$$

$$\beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon}) \rightarrow \xi_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma), \quad (4.31)$$

$$\omega_\varepsilon \rightarrow \omega \quad \text{weakly in } L^2(0, T). \quad (4.32)$$

From (4.27), due a well-known compactness results (see, e.g., [23, Sect. 8, Cor. 4]) we obtain

$$\mathbf{v}_\varepsilon \rightarrow \mathbf{v} \quad \text{strongly in } C([0, T]; \mathbf{H}_0) \cap L^2(0, T; \mathbf{V}_0), \quad (4.33)$$

which also entails

$$\mathbf{u}_\varepsilon \rightarrow \mathbf{u} = \mathbf{v} + m_0 \mathbf{1} \quad \text{strongly in } C([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V}), \quad (4.34)$$

as  $\varepsilon \rightarrow 0$ . We point out that (4.33) implies  $\mathbf{v}(0) = \mathbf{v}_0$ , that is,

$$v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

Moreover, (4.34) and the Lipschitz continuity of  $\pi$  and  $\pi_\Gamma$  ensure that

$$\boldsymbol{\pi}(\mathbf{u}_\varepsilon) \rightarrow \boldsymbol{\pi}(\mathbf{u}) \quad \text{strongly in } C([0, T]; \mathbf{H}),$$

whence (cf. (4.6))  $\omega = m(\boldsymbol{\xi} + \boldsymbol{\pi}(\mathbf{u}) - \mathbf{f})$  with  $\boldsymbol{\xi} = (\xi, \xi_\Gamma)$ . Moreover, by applying [2, Prop. 2.2, p. 38] and using (4.30)–(4.31) with (4.34), we deduce that

$$\xi \in \beta(u) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{a.e. on } \Sigma$$

due to the maximal monotonicity of  $\beta$  and  $\beta_\Gamma$ . At this point, we can pass to the limit in (4.8)–(4.9) obtaining (2.11)–(2.12). Thus, it turns out that the triplet  $(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\xi})$  is a weak solution of (P).

### 4.4 Regularity result

In this subsection, we try to prove a regularity estimate allowing us to fix a strong solution of (P). Then, our third theorem deals with the existence of the strong solution. To this aim, we introduce the following additional regularity assumptions for  $\mathbf{f}$  and  $\mathbf{u}_0$ :

$$(A8) \quad \mathbf{f} \in H^1(0, T; \mathbf{H});$$

$$(A9) \quad \mathbf{u}_0 \in \mathbf{W} \text{ and the family } \{-\partial\varphi(\mathbf{u}_0 - m_0 \mathbf{1}) - \boldsymbol{\beta}_\varepsilon(\mathbf{u}_0) - \boldsymbol{\pi}(\mathbf{u}_0) + \mathbf{f}(0) : \varepsilon \in (0, \varepsilon_0]\} \text{ is bounded in } \mathbf{V} \text{ for some } \varepsilon_0 \in (0, 1].$$

These assumptions, in particular (A9), can be compared with the analogous ones in [12]. Let us point out that (A8) entails the validity of (A1) and (A5).

**Theorem 2.3.** *Under the assumptions (A3), (A4), (A6)–(A9), there exists a strong solution of (P).*

**Proof.** Recall (4.3)–(4.4) at times  $s, s+h \in (0, T)$ . Take the difference of them and test the resultant by  $\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)$  where  $0 < h < T - s$ . Then, by virtue of (A4) and the compactness inequality (3.2), we have that

$$\begin{aligned} & \frac{\varepsilon}{2} \frac{d}{ds} |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{H}_0}^2 + \frac{1}{2} \frac{d}{ds} |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 \\ & \quad + |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0}^2 + (\beta_\varepsilon(\mathbf{u}_\varepsilon(s)) - \beta_\varepsilon(\mathbf{u}_\varepsilon(s+h)), \mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s))_{\mathbf{H}} \\ & \leq (L + L_\Gamma) |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{H}}^2 + \frac{1}{2} |\mathbf{f}_\varepsilon(s+h) - \mathbf{f}_\varepsilon(s)|_{\mathbf{H}}^2 + \frac{1}{2} |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{H}}^2 \\ & \leq \frac{1}{2} |\mathbf{f}_\varepsilon(s+h) - \mathbf{f}_\varepsilon(s)|_{\mathbf{H}}^2 + \tilde{\delta} |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0}^2 + c_{\tilde{\delta}} |\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)|_{\mathbf{V}_0^*}^2 \end{aligned}$$

for a.a.  $s \in (0, T)$ , where  $\tilde{\delta} > 0$  and  $c_{\tilde{\delta}} > 0$  is a constant depending on  $L, L_\Gamma$ . Taking  $\tilde{\delta} = 1/2$ , dividing the resultant by  $h^2$ , integrating it over  $(0, t)$  with respect to  $s$ , and using the monotonicity of  $\beta$ , we infer that

$$\begin{aligned} & \frac{\varepsilon}{2} \left| \frac{\mathbf{v}_\varepsilon(t+h) - \mathbf{v}_\varepsilon(t)}{h} \right|_{\mathbf{H}_0}^2 + \frac{1}{2} \left| \frac{\mathbf{v}_\varepsilon(t+h) - \mathbf{v}_\varepsilon(t)}{h} \right|_{\mathbf{V}_0^*}^2 + \frac{1}{2} \int_0^t \left| \frac{\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)}{h} \right|_{\mathbf{V}_0}^2 ds \\ & \leq \frac{\varepsilon}{2} \left| \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h} \right|_{\mathbf{H}_0}^2 + \frac{1}{2} \left| \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h} \right|_{\mathbf{V}_0^*}^2 \\ & \quad + \frac{1}{2} \int_0^t \left| \frac{\mathbf{f}(s+h) - \mathbf{f}(s)}{h} \right|_{\mathbf{H}}^2 ds + c_{\tilde{\delta}} \int_0^t \left| \frac{\mathbf{v}_\varepsilon(s+h) - \mathbf{v}_\varepsilon(s)}{h} \right|_{\mathbf{V}_0^*}^2 ds. \end{aligned}$$

Now, we need that the first two terms in the right hand side remain bounded, uniformly for all  $h > 0$  sufficiently small. Therefore, we go back to (4.3) and integrate it from 0 to  $h$ , then test by  $(\mathbf{v}_\varepsilon(h) - \mathbf{v}_0)/h^2$  getting

$$\begin{aligned} & \frac{\varepsilon}{2} \left| \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h} \right|_{\mathbf{H}_0}^2 + \frac{1}{2} \left| \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h} \right|_{\mathbf{V}_0^*}^2 \\ & \leq - \left( \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h}, \frac{1}{h} \int_0^h \mathbf{P}(\partial\varphi(\mathbf{v}_\varepsilon(s)) + \beta_\varepsilon(\mathbf{u}_\varepsilon(s)) + \pi(\mathbf{u}_\varepsilon(s)) - \mathbf{f}(s)) ds \right)_{\mathbf{H}_0} \\ & \leq \frac{1}{4} \left| \frac{\mathbf{v}_\varepsilon(h) - \mathbf{v}_0}{h} \right|_{\mathbf{V}_0^*}^2 + \left| \frac{1}{h} \int_0^h (\partial\varphi(\mathbf{v}_\varepsilon(s)) + \beta_\varepsilon(\mathbf{u}_\varepsilon(s)) + \pi(\mathbf{u}_\varepsilon(s)) - \mathbf{f}(s)) ds \right|_{\mathbf{V}}^2, \quad (4.35) \end{aligned}$$

where we have used some properties of the projection operator  $\mathbf{P}$  defined by (2.26), in particular that  $|\mathbf{P}\mathbf{z}|_{\mathbf{V}_0} \leq |\mathbf{z}|_{\mathbf{V}}$  for all  $\mathbf{z} \in \mathbf{V}$ . In order that the last quantity in (4.35) be bounded, we need that  $\mathbf{v}_0 = \mathbf{u}_0 - m_0 \mathbf{1} \in D(\partial\varphi)$  and especially that

$$-\partial\varphi(\mathbf{v}_0) - \beta_\varepsilon(\mathbf{u}_0) - \pi(\mathbf{u}_0) + \mathbf{f}(0)$$

remains bounded in  $\mathbf{V}$  for all  $\varepsilon \in (0, \varepsilon_0]$  (compare with the assumption (2.40) in the paper [12] and see the comments just following (2.40)). Therefore, thanks to (A8) and (A9), the Gronwall inequality implies that the functions

$$t \mapsto \frac{\mathbf{v}_\varepsilon(t+h) - \mathbf{v}_\varepsilon(t)}{h}$$

are bounded in  $L^\infty(0, T-h; \mathbf{V}_0^*) \cap L^2(0, T-h; \mathbf{V}_0)$ , and

$$t \mapsto \varepsilon^{1/2} \frac{\mathbf{v}_\varepsilon(t+h) - \mathbf{v}_\varepsilon(t)}{h}$$

are bounded in  $L^\infty(0, T-h; \mathbf{H}_0)$  uniformly with respect to  $\varepsilon \in (0, \varepsilon_0]$ , so that passing to the limit as  $h \rightarrow 0$  we obtain the bounds

$$\varepsilon^{1/2} |\mathbf{v}'_\varepsilon|_{L^\infty(0, T; \mathbf{H}_0)} + |\mathbf{v}'_\varepsilon|_{L^\infty(0, T; \mathbf{V}_0^*)} + |\mathbf{v}'_\varepsilon|_{L^2(0, T; \mathbf{V}_0)} \leq M_9, \quad (4.36)$$

where  $M_9$  is a positive constant, independent of  $\varepsilon \in (0, \varepsilon_0]$ . From estimate (4.36) it follows that the estimates for  $\{\beta_\varepsilon(u_\varepsilon)\}$  and  $\{\beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon})\}$  in (4.14) can be improved, since the left hand side of (4.3) is now bounded in  $L^\infty(0, T; \mathbf{V}_0^*)$  so that we can test (4.3) by  $\mathbf{v}_\varepsilon$  (bounded in  $L^\infty(0, T; \mathbf{V}_0)$ ) and argue as in (4.16), but without a final integration in time. Proceeding in this way, we obtain

$$|\beta_\varepsilon(u_\varepsilon)|_{L^\infty(0, T; L^1(\Omega))} + |\beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon})|_{L^\infty(0, T; L^1(\Gamma))} \leq M_{10}.$$

Also, the estimates (4.21) can be improved to

$$|\omega_\varepsilon|_{L^\infty(0, T)} + |\boldsymbol{\mu}_\varepsilon|_{L^\infty(0, T; \mathbf{V})} \leq M_{11},$$

since  $\mathbf{v}'_\varepsilon$  is bounded in  $L^\infty(0, T; \mathbf{V}_0^*)$  and  $\mathbf{f} \in L^\infty(0, T; \mathbf{H})$ . At this point, we can proceed analogously in modifying Lemmas 4.4 and 4.5 without making the final integrations over  $(0, T)$  in the proofs, but deducing  $L^\infty$  bounds. In particular we find that

$$\begin{aligned} |\beta_\varepsilon(u_\varepsilon)|_{L^\infty(0, T; H)} &\leq M_{12}, \\ |\partial_\nu v_\varepsilon|_{L^\infty(0, T; H_\Gamma)} &\leq M_{12}, \\ |\beta_{\Gamma, \varepsilon}(u_{\Gamma, \varepsilon})|_{L^\infty(0, T; H_\Gamma)} &\leq M_{12}, \\ |\mathbf{v}_\varepsilon|_{L^\infty(0, T; \mathbf{W})} &\leq M_{12} \end{aligned}$$

in the specified order. Of course,  $M_{10}$ ,  $M_{11}$  and  $M_{12}$  are positive constants independent of  $\varepsilon \in (0, \varepsilon_0]$ . Now, in the subsequent passage to the limit as  $\varepsilon \rightarrow 0$ , for the solution we derive the additional properties that

$$\begin{aligned} \mathbf{v} &\in W^{1, \infty}(0, T; \mathbf{V}_0^*) \cap H^1(0, T; \mathbf{V}_0) \cap L^\infty(0, T; \mathbf{W}), \\ \boldsymbol{\mu} &\in L^\infty(0, T; \mathbf{V}), \quad \boldsymbol{\xi} \in L^\infty(0, T; \mathbf{H}). \end{aligned}$$

Under these regularities, (2.11) can be rewritten as (cf. Remark 2)

$$\begin{aligned} \int_\Omega \partial_t v(t) z dx + \int_\Gamma \partial_t v_\Gamma(t) z_\Gamma d\Gamma + \int_\Omega \nabla \mu(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma \mu_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma &= 0 \\ \text{for all } \mathbf{z} = (z, z_\Gamma) \in \mathbf{V}, \text{ for a.a. } t \in (0, T). \end{aligned} \quad (4.37)$$

Then, taking a test element  $\mathbf{z} = (z, 0)$ , with  $z \in \mathcal{D}(Q)$ , in (4.37) and integrating over  $(0, T)$ , we are led to the equation

$$\partial_t v - \Delta \mu = 0 \quad \text{in } \mathcal{D}'(Q).$$

This implies that  $\Delta \mu \in L^2(0, T; H^1(\Omega))$ , due to the regularity of  $\partial_t v$ . Moreover, thanks to the fact that  $\partial_t u = \partial_t v$ , we also have  $\partial_t u - \Delta \mu = 0$  a.e. in  $Q$ . Furthermore, in view of  $\mu_\Gamma \in L^\infty(0, T; V_\Gamma)$ , we obtain the regularities (see, e.g., [4, Thm. 3.2, p. 1.79] and [4, Thm. 2.25, p. 1.62])

$$\mu \in L^2(0, T; H^{3/2}(\Omega)), \quad \partial_\nu \mu \in L^2(0, T; H_\Gamma)$$

by the trace theorem. The last condition implies that the boundary equation

$$\partial_t u_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{holds true in } L^2(0, T; H_\Gamma) \text{ and a.e. on } \Sigma,$$

and provides additional regularity for  $\mu_\Gamma$ :  $\mu_\Gamma \in L^2(0, T; H^2(\Gamma))$ , which even yields

$$\mu \in L^2(0, T; H^{5/2}(\Omega)). \quad (4.38)$$

Thus, all the regularities for the strong solution specified in Definition 2.2 are finally obtained, with the further regularity (4.38), and equations and conditions (2.14)–(2.18) are satisfied.  $\square$

## 5 Appendix

We use the same notation as in the previous sections for function spaces.

**Lemma A.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be bounded, connected and smooth enough. Then, there exists  $c_p > 0$  such that*

$$c_p |\mathbf{z}|_{\mathbf{V}}^2 \leq |\mathbf{z}|_{\mathbf{V}_0}^2 \quad \text{for all } \mathbf{z} \in \mathbf{V}_0.$$

**Proof.** It suffices to show that there exist  $C_p > 0$  such that

$$|z|_H^2 + |z_\Gamma|_{H_\Gamma}^2 \leq C_p \left( |\nabla z|_{H^d}^2 + |\nabla_\Gamma z_\Gamma|_{H_\Gamma^d}^2 \right) \quad \text{for all } \mathbf{z} := (z, z_\Gamma) \in \mathbf{V}_0.$$

If this is not the case, for any  $n \in \mathbb{N}$  one can find  $\tilde{\mathbf{z}}_n := (\tilde{z}_n, \tilde{z}_{\Gamma,n}) \in \mathbf{V}_0$  such that

$$|\tilde{z}_n|_H^2 + |\tilde{z}_{\Gamma,n}|_{H_\Gamma}^2 > n \left( |\nabla \tilde{z}_n|_{H^d}^2 + |\nabla_\Gamma \tilde{z}_{\Gamma,n}|_{H_\Gamma^d}^2 \right).$$

Taking  $z_n := \tilde{z}_n / (|\tilde{z}_n|_H^2 + |\tilde{z}_{\Gamma,n}|_{H_\Gamma}^2)^{1/2}$ , then  $(z_n)_\Gamma = \tilde{z}_{\Gamma,n} / (|\tilde{z}_n|_H^2 + |\tilde{z}_{\Gamma,n}|_{H_\Gamma}^2)^{1/2}$ . Therefore, if we set  $\mathbf{z}_n := (z_n, z_{\Gamma,n})$  with  $z_{\Gamma,n} := (z_n)_\Gamma$ , we see that  $\mathbf{z}_n \in \mathbf{V}_0$  and

$$1 = |z_n|_H^2 + |z_{\Gamma,n}|_{H_\Gamma}^2 > n \left( |\nabla z_n|_{H^d}^2 + |\nabla_\Gamma z_{\Gamma,n}|_{H_\Gamma^d}^2 \right) \quad \text{for all } n \in \mathbb{N}; \quad (5.1)$$

hence, we have

$$|z_n|_H^2 \leq 1, \quad |z_{\Gamma,n}|_{H_\Gamma}^2 \leq 1, \quad |\nabla z_n|_{H^d}^2 \leq \frac{1}{n}, \quad |\nabla_\Gamma z_{\Gamma,n}|_{H_\Gamma^d}^2 \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

These inequalities imply that there exist a subsequence of  $n$  (not relabeled) and some limit function  $\mathbf{z} := (z, z_\Gamma) \in \mathbf{V}$  such that

$$z_n \rightarrow z \quad \text{weakly in } V, \text{ strongly in } H, \quad (5.2)$$

$$z_{\Gamma,n} \rightarrow z_\Gamma \quad \text{weakly in } V_\Gamma, \text{ strongly in } H_\Gamma \quad (5.3)$$

as  $n \rightarrow \infty$ ,

$$\nabla z = 0 \quad \text{a.e. in } \Omega, \quad \nabla_\Gamma z_\Gamma = 0 \quad \text{a.e. on } \Gamma, \quad (5.4)$$

$$\int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma = 0; \quad (5.5)$$

here the property (5.5) implies that  $\mathbf{z} \in \mathbf{V}_0$ . Now, since  $\Omega$  is connected condition (5.4) entails that  $z$  and  $z_\Gamma$  are the same constant function. From (5.1)–(5.3) we have  $|\mathbf{z}|_{\mathbf{H}_0} = 1$ , that is,  $z \neq 0$  and  $z_\Gamma \neq 0$ , however in contradiction with (5.5). This completes the proof of Lemma A.  $\square$

**Lemma B.**  $\mathbf{V}_0$  is densely and compactly embedded into  $\mathbf{H}_0$ .

**Proof.** The strategy of the proof is essentially same as in [10, Prop. A1.1]. For a fixed  $\mathbf{z} = (z, z_\Gamma) \in \mathbf{H}_0$  and for  $n \in \mathbb{N}$ , consider the following elliptic problem:

$$z_n - \frac{1}{n} \Delta z_n = z \quad \text{a.e. in } \Omega, \quad (5.6)$$

$$\frac{1}{n} \partial_\nu z_n + (z_n)|_\Gamma = z_\Gamma \quad \text{a.e. on } \Gamma. \quad (5.7)$$

Is is not difficult to check that (see, e.g., [10, Prop. A1.1]) this sequence  $\mathbf{z}_n := (z_n, z_{\Gamma,n})$ , with  $z_{\Gamma,n} := (z_n)|_\Gamma$ , satisfies

$$\mathbf{z}_n \rightarrow \mathbf{z} \quad \text{strongly in } \mathbf{H} \quad \text{as } n \rightarrow +\infty.$$

Moreover, thanks to  $\mathbf{z} \in \mathbf{H}_0$  we obtain

$$\begin{aligned} \int_\Omega z_n dx + \int_\Gamma z_{\Gamma,n} d\Gamma &= \int_\Omega \left( z + \frac{1}{n} \Delta z_n \right) dx + \int_\Gamma \left( z_\Gamma - \frac{1}{n} \partial_\nu z_n \right) d\Gamma \\ &= \left( \int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma \right) + \frac{1}{n} \left( \int_\Omega \operatorname{div} \nabla z_n dx - \int_\Gamma \partial_\nu z_n d\Gamma \right) = 0. \end{aligned}$$

This means that  $\{\mathbf{z}_n\}_{n \in \mathbb{N}} \subset \mathbf{H}_0$ , that is,  $\mathbf{V}_0$  is dense in  $\mathbf{H}_0$ . Next, from the compact embeddings  $V \subset H$  and  $V_\Gamma \subset H$ , we easily see that for any bounded sequence  $\{\mathbf{z}_n\}_{n \in \mathbb{N}}$  in  $\mathbf{V}_0$ , the sequences of each component,  $\{z_n\}_{n \in \mathbb{N}}$  and  $\{z_{\Gamma,n}\}_{n \in \mathbb{N}}$ , have common subsequences that converge strongly to some limit functions  $z$  and  $z_\Gamma$ , respectively. Therefore, we complete the proof of Lemma B.  $\square$

**Lemma C.** *Let  $\varphi : \mathbf{H}_0 \rightarrow [0, +\infty]$  be defined by (4.23). Then the subdifferential  $\partial\varphi$  on  $\mathbf{H}_0$  is characterized by*

$$\partial\varphi(\mathbf{z}) = (-\Delta z, \partial_\nu z - \Delta_\Gamma z_\Gamma) \quad \text{with} \quad \mathbf{z} = (z, z_\Gamma) \in D(\partial\varphi) = \mathbf{W} \cap \mathbf{V}_0.$$

**Proof.** Let  $\mathbf{z}^* := (z^*, z_\Gamma^*) \in \partial\varphi(\mathbf{z})$  in  $\mathbf{H}_0$ . Then, from the definition of subdifferential it is straightforward to obtain

$$(\mathbf{z}^*, \tilde{\mathbf{z}})_{\mathbf{H}_0} = (\nabla z, \nabla \tilde{z})_{H^d} + (\nabla_\Gamma z_\Gamma, \nabla_\Gamma \tilde{z}_\Gamma)_{H_\Gamma^d} \quad \text{for all } \tilde{\mathbf{z}} := (\tilde{z}, \tilde{z}_\Gamma) \in \mathbf{V}_0. \quad (5.8)$$

For each  $\tilde{z} \in \mathcal{D}(\Omega)$ , put  $\tilde{\mathbf{z}} = (\tilde{z}, 0)$  and take the test function  $\mathbf{P}\tilde{\mathbf{z}} \in \mathbf{V}_0$  above. Using the property (2.27) we infer that

$$\begin{aligned} \int_\Omega z^* \tilde{z} dx &= (\mathbf{z}^*, \tilde{\mathbf{z}})_{\mathbf{H}} = (\mathbf{z}^*, \mathbf{P}\tilde{\mathbf{z}})_{\mathbf{H}_0} \\ &= (\nabla z, \nabla(\tilde{z} - m(\tilde{\mathbf{z}})))_{H^d} + (\nabla_\Gamma z_\Gamma, \nabla_\Gamma(0 - m(\tilde{\mathbf{z}})))_{H_\Gamma^d} = (\nabla z, \nabla \tilde{z})_{H^d}, \end{aligned}$$

namely,  $z^* = -\Delta z$  in  $\mathcal{D}'(\Omega)$ . Now, as  $\mathbf{z} = (z, z_\Gamma) \in D(\varphi) = \mathbf{V}_0$ , we have that the trace  $z|_\Gamma = z_\Gamma$  is in  $V_\Gamma$  and  $-\Delta z = z^*$  lies in  $H$ . Hence, from the theory for elliptic regularity (see, e.g., [4, Thm. 3.2, p. 1.79]), it follows that  $z \in H^{3/2}(\Omega)$ . In turn, the trace theory implies that  $\partial_\nu z \in H_\Gamma$  (cf. [4, Thm. 2.25, p. 1.62]). Then, recalling (5.8), for a general test function  $\tilde{\mathbf{z}} \in \mathbf{V}_0$  we have that

$$\begin{aligned} (\mathbf{z}^*, \tilde{\mathbf{z}})_{\mathbf{H}_0} &= (z^*, \tilde{z})_H + (z_\Gamma^*, \tilde{z}_\Gamma)_{H_\Gamma} \\ &= (\nabla z, \nabla \tilde{z})_{H^d} + (\nabla_\Gamma z_\Gamma, \nabla_\Gamma \tilde{z}_\Gamma)_{H_\Gamma^d} \\ &= -(\Delta z, \tilde{z})_H + (\partial_\nu z, \tilde{z}_\Gamma)_{H_\Gamma} + (\nabla_\Gamma z_\Gamma, \nabla_\Gamma \tilde{z}_\Gamma)_{H_\Gamma^d}, \end{aligned}$$

whence, being  $-\Delta z = z^*$  in  $H$ , we deduce that

$$(z_\Gamma^* - \partial_\nu z, \tilde{z}_\Gamma)_{H_\Gamma} = (\nabla_\Gamma z_\Gamma, \nabla_\Gamma \tilde{z}_\Gamma)_{H_\Gamma^d}.$$

The last equality implies that  $-\Delta_\Gamma z_\Gamma = z_\Gamma^* - \partial_\nu z$  is in  $H_\Gamma$  and consequently  $z_\Gamma \in H^2(\Gamma)$  follows from the boundary version of the elliptic regularity theory. Since  $z_\Gamma$  is the trace of  $z$  on the boundary  $\Gamma$  and is sufficiently smooth (indeed,  $z_\Gamma \in H^{3/2}(\Gamma)$ ), by the elliptic regularity again we conclude that  $z \in H^2(\Omega)$ , which finally leads to  $D(\partial\varphi) = \mathbf{W} \cap \mathbf{V}_0$ .  $\square$

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